

Invariant Discretization of Partial Differential Equations Admitting Infinite-Dimensional Symmetry Groups

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Abstract

This paper is concerned with the invariant discretization of differential equations admitting infinite-dimensional symmetry groups. By way of example, we first show that there are differential equations with infinite-dimensional symmetry groups that do not admit enough joint invariants preventing the construction of invariant finite difference approximations. To solve this shortage of joint invariants we propose to discretize the pseudo-group action. Computer simulations indicate that the numerical schemes constructed from the joint invariants of discretized pseudo-group can produce better numerical results than standard schemes.

1 Introduction

For the last 20 years, a considerable amount of work has been invested into the problem of invariantly discretizing differential equations with symmetries. This effort is part of a larger program aiming to extend Lie's theory of transformation groups to finite difference equations, [16]. With the emergence of physical models based on discrete spacetime, and in light of the importance of symmetry in our understanding of modern physics, the problem of invariantly discretizing differential equations is still of present interest. From a theoretical standpoint, working with invariant numerical schemes allows one to use standard Lie group techniques to find explicit solutions, [27], or compute conservation laws, [8]. From a more practical point of view, the motivation stems from the fact that invariant schemes have been shown to outperform standard numerical methods in a number of examples, [2, 6, 12, 26].

In general, to build an invariant numerical scheme one has to construct joint invariants (also known as finite difference invariants). These joint invariants are usually found using one of two methods. One can either use Lie's method of infinitesimal generators which requires solving a system of linear partial differential equations, [7, 16], or

the method of equivariant moving frames which requires solving a system of (nonlinear) algebraic equations, [12, 20]. Both approaches produce joint invariants which, in the coalescent limit, converge to differential invariants of the prolonged action. Thus far, the theory and applications found in the literature primarily deal with finite-dimensional Lie group actions and the case of infinite-dimensional Lie pseudo-groups as yet to be satisfactorily treated. Many partial differential equations in hydrodynamics or meteorology admit infinite-dimensional symmetry groups. The Navier–Stokes equation, [18], the Kadomtsev–Petviashvili equation, [5], and the Davey–Stewartson equations, [4], are classical examples of such equations. Linear or linearizable partial differential equations also form a large class of equations admitting infinite-dimensional symmetry groups.

To construct invariant numerical schemes of differential equations admitting symmetries, one of the main steps consists of finding joint invariants that approximate the differential invariants of the symmetry group. For finite-dimensional Lie group actions, this can always be done by considering the product action on sufficiently many points. Unfortunately, as the next example shows, the same is not true for infinite-dimensional Lie pseudo-group actions.

Example 1.1. Let $f(x) \in \mathcal{D}(\mathbb{R})$ be a local diffeomorphism of \mathbb{R} . Throughout the paper we will use the infinite-dimensional pseudo-group

$$X = f(x), \quad Y = y, \quad U = \frac{u}{f'(x)}, \quad (1.1)$$

acting on $\mathbb{R}^3 \setminus \{u = 0\}$, to illustrate the theory and constructions. The pseudo-group (1.1) was introduced by Lie, [15, p.373], in his study of second order partial differential equations integrable by the method of Darboux. It also appears in Vessiot’s work on group splitting and automorphic systems, [29], in Kumpera’s investigation of Lie’s theory of differential invariants based on Spencer’s cohomology, [13], and recently in [21, 22, 25] to illustrate a new theoretical foundation of moving frames.

The differential invariants of the pseudo-group action (1.1) can be found in [22]. One of these invariants is

$$I_{1,1} = \frac{u u_{xy} - u_x u_y}{u^3}. \quad (1.2)$$

With (1.2) it is possible to form the partial differential equation

$$\frac{u u_{xy} - u_x u_y}{u^3} = 1, \quad (1.3)$$

which was used in [25] to illustrate the method of symmetry reduction of exterior differential systems.

By construction, Equation (1.3) is invariant under the pseudo-group¹ (1.1). To obtain an invariant discretization of (1.3), an invariant approximation of the differential invariant (1.2) must be found. To discretize the invariant (1.2), the multi-index $(m, n) \in \mathbb{Z}^2$ is introduced to label sample points:

$$x_{m,n}, \quad y_{m,n}, \quad u_{m,n} = u(x_{m,n}, y_{m,n}). \quad (1.4)$$

¹Equation (1.3) admits a larger symmetry group given by $X = f(x)$, $Y = g(y)$, $U = u/(f'(x)g'(y))$, with $f, g \in \mathcal{D}(\mathbb{R})$. This pseudo-group is considered in Example 3.20.

Following the general philosophy, [7, 12, 16, 20], the pseudo-group (1.1) induces the product action

$$X_{m,n} = f(x_{m,n}), \quad Y_{m,n} = y_{m,n}, \quad U_{m,n} = \frac{u_{m,n}}{f'(x_{m,n})} \quad (1.5)$$

on the discrete points (1.4). On an arbitrary finite set of points, we claim that the only joint invariants are

$$Y_{m,n} = y_{m,n}. \quad (1.6)$$

To see this, let \mathcal{N} be a finite subset of \mathbb{Z}^2 , and assume $x_{m,n} \in \text{dom } f$ for $(m,n) \in \mathcal{N}$. Since the components $x_{m,n}$ are generically distinct and $f \in \mathcal{D}(\mathbb{R})$ is an arbitrary local diffeomorphism, the *pseudo-group parameters*

$$f(x_{m,n}) \quad \text{and} \quad f'(x_{m,n}) \quad \text{with} \quad (m,n) \in \mathcal{N} \quad (1.7)$$

are independent. Hence, as shown in [10], the pseudo-group (1.5) shares the same invariants as its Lie completion

$$X_{m,n} = f_{m,n}(x_{m,n}), \quad Y_{m,n} = y_{m,n}, \quad U_{m,n} = \frac{u_{m,n}}{f'_{m,n}(x_{m,n})}, \quad (1.8)$$

where for each different subscript $(m,n) \in \mathcal{N}$, the functions $f_{m,n} \in \mathcal{D}(\mathbb{R})$ are functionally independent local diffeomorphisms². For the Lie completion (1.8), it is clear that (1.6) are the only admissible invariants. Hence, generically, we conclude that it is not possible to approximate the differential invariant (1.2) by joint invariants.

To construct additional joint invariants, invariant constraints on the independent variables $x_{m,n}$ need to be imposed to reduce the number of pseudo-group parameters (1.7). To reduce this number as much as possible, we assume that

$$x_{m,n+1} = x_{m,n}. \quad (1.9)$$

Equation (1.9) is seen to be invariant under the product action (1.5) since

$$X_{m,n+1} = f(x_{m,n+1}) = f(x_{m,n}) = X_{m,n}$$

when (1.9) holds. Equation (1.9) implies that $x_{m,n}$ is independent of the index n :

$$x_{m,n} = x_m.$$

To cover (a region of) the xy -plane,

$$\Delta x_m = x_{m+1} - x_m \neq 0 \quad \text{and} \quad \delta y_{m,n} = y_{m,n+1} - y_{m,n} \neq 0$$

must hold. Since the variables $y_{m,n}$ are invariant under the product action (1.5) we can, for simplicity, set

$$y_{m,n} = y_n = k n + y_0, \quad (1.10)$$

where $k > 0$ and y_0 are constants. To respect the product action (1.5) we cannot require the step size $\Delta x_m = x_{m+1} - x_m$ to be constant as this is not an invariant assumption

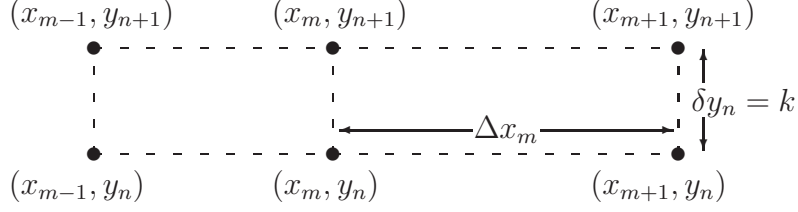


Figure 1: Rectangular mesh.

of the pseudo-group action. Thus, in general, the mesh in the independent variables (x_m, y_n) will be rectangular with variable step sizes in x , see Figure 1.

Repeating the argument above, when (1.9) and (1.10) hold, the joint invariants of the product action (1.5) are

$$y_n, \quad \frac{u_{m,n+k}}{u_{m,n}}, \quad k, m, n \in \mathbb{Z}. \quad (1.11)$$

Introducing the dilation group

$$X = x, \quad Y = y, \quad U = \lambda u, \quad \lambda > 0, \quad (1.12)$$

we see that the differential invariant (1.2) cannot be approximated by the joint invariants (1.11). Indeed, since the invariants $u_{m,n+k}/u_{m,n}$ are homogeneous of degree 0, any combination of the invariants (1.11) will converge to a differential invariant of homogeneous degree 0. On the other hand, the differential invariant (1.2) is homogeneous of degree -1 under (1.12).

As it stands, it is not possible to construct joint invariants that approximate the differential invariant (1.2). To remedy the problem, one possibility is to reduce the size of the symmetry group by considering sub-pseudo-groups. For the diffeomorphism pseudo-group $\mathcal{D}(\mathbb{R})$, since the largest non-trivial sub-pseudo-group is the special linear group $SL(2)$, [19], this approach drastically changes the nature of the action as it transitions from an infinite-dimensional transformation group to a three-dimensional group of transformations. In this paper we are interested in preserving the infinite-dimensional nature of transformation groups and propound another suggestion. Taking the point of view that the notion of derivative is not defined in the discrete setting, we propose to discretize infinite-dimensional pseudo-group actions. In other words, derivatives are to be replaced by finite difference approximations. For the pseudo-group (1.1), instead of considering the product action (1.5), we suggest to work with the first order approximation

$$X_m = f(x_m), \quad Y_n = y_n = k n + y_0, \quad U_{m,n} = u_{m,n} \cdot \frac{x_{m+1} - x_m}{f(x_{m+1}) - f(x_m)}. \quad (1.13)$$

²It is customary to use the notation $f_{m,n} = f(x_{m,n})$ to denote the value of the function $f(x)$ at the point $x_{m,n}$, and this is the convention used in Sections 3, 4, and 5. In equation (1.8), the subscript attached to the diffeomorphism $f_{m,n}(x_{m,n})$ has a different meaning. Here, the subscript (m,n) is used to denote different diffeomorphisms. Thus, the pseudo-group (1.5) is contained in the Lie completion (1.8). This particular use of the subscript only occurs in (1.8).

In Section 3, joint invariants of the pseudo-group action (1.13) are constructed and an invariant numerical scheme approximating (1.3) is obtained in Section 4.

To develop our ideas we opted to use the theory of equivariant moving frames, [20, 22], but our constructions can also be recast within Lie's infinitesimal framework. In Section 2, the concept of an infinite-dimensional Lie pseudo-group is recalled and the equivariant moving frame construction is summarized. In Section 3, pseudo-group actions are discretized and the equivariant moving frame construction is adapted to those actions. Along with (1.1), the pseudo-group

$$X = f(x), \quad Y = y f'(x) + g(x), \quad U = u + \frac{y f''(x) + g'(x)}{f'(x)}, \quad (1.14)$$

with $f \in \mathcal{D}(\mathbb{R})$ and $g \in C^\infty(\mathbb{R})$, will stand as a second example to illustrate our constructions. We choose to work with the pseudo-groups (1.1) and (1.14) to keep our examples relatively simple. Furthermore, these pseudo-groups have been extensively used in [21, 22, 23, 24] to illustrate the (continuous) method of moving frames. With these well-documented examples, it allowed us to verify that our discrete constructions and computations did converge to their continuous counterparts.

Finally, in Section 5 an invariant numerical approximation of (1.3) is compared to a standard discretization of the equation. Our numerical tests show that the invariant scheme is more precise and stable than the standard scheme.

2 Lie Pseudo-groups and moving frames

For completeness, we begin by recalling the definition of a pseudo-group, [3, 13, 14, 21, 22, 28]. Let M be an m -dimensional manifold. By a local diffeomorphism of M we mean a one-to-one map $\varphi: U \rightarrow V$ defined on open subsets $U, V = \varphi(U) \subset M$, with inverse $\varphi^{-1}: V \rightarrow U$.

Definition 2.1. A collection \mathcal{G} of local diffeomorphisms of M is a *pseudo-group* if

- \mathcal{G} is closed under restriction: if $U \subset M$ is an open set and $g: U \rightarrow M$ is in \mathcal{G} , then so is the restriction $g|_V$ for all open $V \subset U$.
- Elements of \mathcal{G} can be pieced together: if $U_\nu \subset M$ are open subsets, $U = \bigcup_\nu U_\nu$, and $g: U \rightarrow M$ is a local diffeomorphism with $g|_{U_\nu} \in \mathcal{G}$ for all ν , then $g \in \mathcal{G}$.
- \mathcal{G} contains the identity diffeomorphism: $1 \cdot z = z$ for all $z \in M = \text{dom } 1$.
- \mathcal{G} is closed under composition: if $g: U \rightarrow M$ and $h: V \rightarrow M$ are two diffeomorphisms belonging to \mathcal{G} , and $g(U) \subset V$, then $h \cdot g \in \mathcal{G}$.
- \mathcal{G} is closed under inversion: if $g: U \rightarrow M$ is in \mathcal{G} then so is $g^{-1}: g(U) \rightarrow M$.

Example 2.2. One of the simplest pseudo-group is given by the collection of local diffeomorphisms $\mathcal{D} = \mathcal{D}(M)$ of a manifold M . All other pseudo-groups defined on M are sub-pseudo-groups of \mathcal{D} .

For $0 \leq n \leq \infty$, let $\mathcal{D}^{(n)} = J^{(n)}(M, M)$ denote the bundle formed by the n^{th} order jets of local diffeomorphisms of M . Local coordinates on $\mathcal{D}^{(n)}$ are given by $\varphi^{(n)}|_z = (z, Z^{(n)})$, where $z = (z^1, \dots, z^m)$ are the source coordinates of the local diffeomorphism, $Z = \varphi(z)$, $Z = (Z^1, \dots, Z^m)$ its target coordinates, and $Z^{(n)}$ collects the derivatives of the target coordinates Z^a with respect to the source coordinates z^b of order $\leq n$. For $k \geq n$, the standard projection is denoted $\tilde{\pi}_n^k: \mathcal{D}^{(k)} \rightarrow \mathcal{D}^{(n)}$.

Definition 2.3. A pseudo-group $\mathcal{G} \subset \mathcal{D}$ is called a *Lie pseudo-group* of order $n^* \geq 1$ if, for all finite $n \geq n^*$:

- $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ forms a smooth embedded subbundle,
- the projection $\tilde{\pi}_n^{n+1}: \mathcal{G}^{(n+1)} \rightarrow \mathcal{G}^{(n)}$ is a fibration,
- every local diffeomorphism $g \in \mathcal{D}$ satisfying $g^{(n^*)} \subset \mathcal{G}^{(n^*)}$ belongs to \mathcal{G} ,
- $\mathcal{G}^{(n)} = \text{pr}^{(n-n^*)}\mathcal{G}^{(n^*)}$ is obtained by prolongation.

In local coordinates, the subbundle $\mathcal{G}^{(n^*)} \subset \mathcal{D}^{(n^*)}$ is characterized by a system of n^{th} order (formally integrable) partial differential equations

$$F^{(n^*)}(z, Z^{(n^*)}) = 0, \quad (2.1)$$

called the n^{th} order *determining system* of the pseudo-group. A Lie pseudo-group is said to be of *finite type* if the solution space of (2.1) only involves a finite number of arbitrary constants. Lie pseudo-groups of finite type are thus isomorphic to local Lie group actions. On the other hand, a Lie pseudo-group is of *infinite type* if it involves arbitrary functions.

Remark 2.4. Linearizing (2.1) at the identity jet $\mathbb{1}^{(n^*)}$ yields the *infinitesimal determining equations*

$$L^{(n^*)}(z, \zeta^{(n^*)}) = 0 \quad (2.2)$$

for an infinitesimal generator

$$\mathbf{v} = \sum_{a=1}^m \zeta^a(z) \frac{\partial}{\partial z^a}. \quad (2.3)$$

The vector field (2.3) is in the Lie algebra \mathfrak{g} of infinitesimal generators of \mathcal{G} if its components are solution of (2.2). Given a differential equation $\Delta(x, u^{(n)}) = 0$ with symmetry group \mathcal{G} , the infinitesimal determining system (2.2) is equivalent to the equations obtained by Lie's standard algorithm for determining the symmetry algebra of the differential equation $\Delta = 0$, [18].

Example 2.5. The pseudo-group (1.1) is a Lie pseudo-group. The first order determining equations are

$$\begin{aligned} X_y = X_u = 0, \quad Y = y, \quad Y_x = Y_u = 0, \quad Y_y = 1, \\ UX_x = u, \quad U_x X_x = 1, \quad U_y = 0. \end{aligned} \quad (2.4)$$

If

$$\mathbf{v} = \xi(x, y, u) \frac{\partial}{\partial x} + \eta(x, y, u) \frac{\partial}{\partial y} + \varphi(x, y, u) \frac{\partial}{\partial u}$$

denotes a (local) vector fields in $\mathbb{R}^3 \setminus \{u = 0\}$, the linearization of (2.4) at the identity jet yields the first order infinitesimal determining equations

$$\xi_y = \xi_u = 0, \quad \eta = \eta_x = \eta_y = \eta_u = 0, \quad \varphi = -u\xi_x, \quad \varphi_y = 0, \quad \varphi_u = -\xi_x.$$

The general solution to this system of equations is

$$\mathbf{v} = a(x) \frac{\partial}{\partial x} - u a'(x) \frac{\partial}{\partial u},$$

where $a(x)$ is an arbitrary smooth function.

Given a Lie pseudo-group \mathcal{G} acting on M , we are now interested in the induced action on p -dimensional submanifolds $S \subset M$ with $1 \leq p < m = \dim M$. It is customary to introduce adapted coordinates

$$z = (x, u) = (x^1, \dots, x^p, u^1, \dots, u^q) \quad (2.5)$$

on M so that, locally, a submanifold S transverse to the vertical fibre $\{x = x_0\}$ is given as the graph of a function $S = \{(x, u(x))\}$. For each integer $0 \leq n \leq \infty$, let $J^{(n)} = J^{(n)}(M, p)$ denote the n^{th} order *submanifold jet bundle* defined as the set of equivalence classes under the equivalence relation of n^{th} order contact, [19]. For $k \geq n$, let $\pi_n^k: J^{(k)} \rightarrow J^{(n)}$ denote the canonical projection. In the adapted system of coordinates $z = (x, u)$, coordinates on $J^{(n)}$ are given by

$$z^{(n)} = (x, u^{(n)}) = (\dots x^i \dots u_{x^J}^\alpha \dots), \quad (2.6)$$

where $u^{(n)}$ denotes the collection of derivatives $u_{x^J}^\alpha$ of order $0 \leq \#J \leq n$.

Alternatively, when no distinction between dependent and independent variables is made, a submanifold $S \subset M$ can be locally parameterized by p variables $s = (s^1, \dots, s^p) \in \mathbb{R}^p$ so that

$$z(s) = (x(s), u(s)) \in S.$$

In the numerical analysis community, the variables $s = (s^1, \dots, s^p)$ are called *computational variables*, [9]. We let $\mathcal{J}^{(n)}$ denote the n^{th} order jet space of submanifolds $S \subset M$ parametrized by computational variables. Local coordinates on $\mathcal{J}^{(n)}$ are given by

$$\mathfrak{z}^{(n)} = (s, x^{(n)}, u^{(n)}) = (\dots s^i \dots x_{s^A}^i \dots u_{s^A}^\alpha \dots), \quad (2.7)$$

with $1 \leq i \leq p$, $1 \leq \alpha \leq q$, and $0 \leq \#A \leq n$. The transition between the jet coordinates (2.6) and (2.7) is given by the chain rule. Provided

$$\det\left(\frac{\partial x^i}{\partial s^j}\right) \neq 0, \quad (2.8)$$

successive application of the implicit total differential operators

$$D_{x^i} = \sum_{j=1}^p W_i^j D_{s^j}, \quad (W_i^j) = (x_{s^i}^j)^{-1}, \quad (2.9)$$

to the dependent variables u^α will give the coordinate expressions for the x derivatives of u in terms of the s derivatives of x and u :

$$u_{x^J}^\alpha = D_{x^{j_1}} \cdots D_{x^{j_k}} u^\alpha = \left(\sum_{\ell=1}^p W_{j_1}^\ell D_{s^\ell} \right) \cdots \left(\sum_{\ell=1}^p W_{j_k}^\ell D_{s^\ell} \right) u^\alpha. \quad (2.10)$$

Given a Lie pseudo-group \mathcal{G} acting on M , the action is prolonged to the computational variables by requiring that they remain unchanged:

$$g \cdot (s, z) = (s, g \cdot z) \quad \text{for all} \quad g \in \mathcal{G}.$$

By abuse of notation we still use \mathcal{G} to denote the extended action $\{\mathbf{1}\} \times \mathcal{G}$ on $\mathbb{R}^p \times M$.

The complete theory of moving frames for infinite-dimensional Lie pseudo-groups can be found in [22]. For reasons that will become more apparent in the next section

we recall the main constructions over the jet bundle $\mathcal{J}^{(n)}$ rather than $\mathbf{J}^{(n)}$. Using (2.10) one can translate the constructions from $\mathcal{J}^{(n)}$ to $\mathbf{J}^{(n)}$. Let

$$\mathcal{B}^{(n)} = \mathcal{J}^{(n)} \times_M \mathcal{G}^{(n)} \quad (2.11)$$

denote the n^{th} order *lifted bundle*. Local coordinates on $\mathcal{B}^{(n)}$ are given by $(\mathfrak{z}^{(n)}, g^{(n)})$, where the base coordinates are the submanifold jet coordinates $\mathfrak{z}^{(n)} = (s, x^{(n)}, u^{(n)}) \in \mathcal{J}^{(n)}$ and the fibre coordinates are the pseudo-group parameters $g^{(n)}$ where $(x, u) \in \text{dom } g$. A local diffeomorphism $h \in \mathcal{G}$ acts on $\mathcal{B}^{(n)}$ by right multiplication:

$$R_h(\mathfrak{z}^{(n)}, g^{(n)}) = (h^{(n)} \cdot \mathfrak{z}^{(n)}, g^{(n)} \cdot (h^{(n)})^{-1}), \quad (2.12)$$

where defined. The second component of (2.12) corresponds to the usual right multiplication $R_h(g^{(n)}) = g^{(n)} \cdot (h^{(n)})^{-1}$ of the pseudo-group onto $\mathcal{G}^{(n)}$, [22]. The first component $h^{(n)} \cdot \mathfrak{z}^{(n)} = (s, h^{(n)} \cdot x^{(n)}, h^{(n)} \cdot u^{(n)}) = (s, X^{(n)}, U^{(n)})$ is the prolonged action of the pseudo-group \mathcal{G} onto the jet space $\mathcal{J}^{(n)}$. Coordinate expressions for the prolonged action are obtained by differentiating the target coordinates $Z = (X, U)$ with respect to the computational variables s :

$$X_A^i = D_{s^1}^{a^1} \dots D_{s^p}^{a^p} X^i, \quad U_A^\alpha = D_{s^1}^{a^1} \dots D_{s^p}^{a^p} U^\alpha, \quad (2.13)$$

where $A = (a^1, \dots, a^p)$. The expressions (2.13) are invariant under the *lifted action* (2.12) and these functions are called *lifted invariants*.

Definition 2.6. A (*right*) *moving frame* of order n is a \mathcal{G} -equivariant section $\widehat{\rho}^{(n)}$ of the lifted bundle $\mathcal{B}^{(n)} \rightarrow \mathcal{J}^{(n)}$.

In local coordinates, the notation

$$\widehat{\rho}^{(n)}(\mathfrak{z}^{(n)}) = (\mathfrak{z}^{(n)}, \rho^{(n)}(\mathfrak{z}^{(n)}))$$

is used to denote an order n right moving frame. Right equivariance means that for $g \in \mathcal{G}$

$$R_g \widehat{\rho}^{(n)}(\mathfrak{z}^{(n)}) = \widehat{\rho}^{(n)}(g^{(n)} \cdot \mathfrak{z}^{(n)}),$$

where defined.

Definition 2.7. Let

$$\mathcal{G}_{\mathfrak{z}^{(n)}}^{(n)} = \left\{ g^{(n)} \in \mathcal{G}^{(n)}|_{\mathfrak{z}} : g^{(n)} \cdot \mathfrak{z}^{(n)} = \mathfrak{z}^{(n)} \right\}$$

denote the *isotropy subgroup* of $\mathfrak{z}^{(n)} \in \mathcal{J}^{(n)}$. The pseudo-group \mathcal{G} is said to act *freely* at $\mathfrak{z}^{(n)} \in \mathcal{J}^{(n)}$ if $\mathcal{G}_{\mathfrak{z}^{(n)}}^{(n)} = \{\mathbb{1}^{(n)}|_{\mathfrak{z}}\}$. The pseudo-group \mathcal{G} is said to act *freely at order n* if it acts freely on an open subset $\mathcal{V}^{(n)} \subset \mathcal{J}^{(n)}$, called the set of *regular n -jets*.

Theorem 2.8. Suppose \mathcal{G} acts freely on $\mathcal{V}^{(n)} \subset \mathcal{J}^{(n)}$, with its orbits forming a regular foliation. Then an n^{th} order moving frame exists in a neighbourhood of $\mathfrak{z}^{(n)} \in \mathcal{V}^{(n)}$.

Once a pseudo-group acts freely, a result known as the *persistence of freeness*, [23, 24], guarantees that the action remains free under prolongation.

Theorem 2.9. If a Lie pseudo-group \mathcal{G} acts freely at $\mathfrak{z}^{(n)}$, then it acts freely at any $\mathfrak{z}^{(k)} \in \mathcal{J}^{(k)}$, $k \geq n$, with $\pi_n^k(\mathfrak{z}^{(k)}) = \mathfrak{z}^{(n)}$.

Remark 2.10. Theorems 2.8 and 2.9 also hold when the pseudo-group action is locally free, meaning that the isotropy group $\mathcal{G}_{\mathfrak{z}^{(n)}}^{(n)}$ is a discrete subgroup of $\mathcal{G}^{(n)}|_{\mathfrak{z}}$.

An order $n \geq n^*$ moving frame is constructed through a normalization procedure based on the choice of a cross-section $\mathcal{K}^{(n)} \subset \mathcal{V}^{(n)}$ to the pseudo-group orbits. The associated (locally defined) right moving frame section $\hat{\rho}^{(n)}: \mathcal{V}^{(n)} \rightarrow \mathcal{B}^{(n)}$ is uniquely characterized by the condition that $\rho^{(n)}(\mathfrak{z}^{(n)}) \cdot \mathfrak{z}^{(n)} \in \mathcal{K}^{(n)}$. In coordinates, assuming that

$$\mathcal{K}^{(n)} = \{z_{i_1} = c_1, \dots, z_{i_{r_n}} = c_{r_n} : r_n = \dim \mathcal{G}^{(n)}|_{\mathfrak{z}}\} \quad (2.14)$$

is a coordinate cross-section, the moving frame $\hat{\rho}^{(n)}$ is obtained by solving the *normalization equations*

$$Z_{i_1}(s, x^{(n)}, u^{(n)}, g^{(n)}) = c_1, \quad \dots \quad Z_{i_{r_n}}(s, x^{(n)}, u^{(n)}, g^{(n)}) = c_{r_n},$$

for the pseudo-group parameters $g^{(n)} = \rho^{(n)}(\mathfrak{z}^{(n)})$. As one increases the order from n to $k > n$, a new cross-section $\mathcal{K}^{(k)} \subset \mathcal{J}^{(k)}$ must be selected. These cross-sections are required to be compatible meaning that $\pi_n^k(\mathcal{K}^{(k)}) = \mathcal{K}^{(n)}$ for all $k > n$. This in turn, implies the compatibility of the moving frames: $\hat{\pi}_n^k \circ \hat{\rho}^{(k)} = \hat{\rho}^{(n)} \circ \pi_n^k$, where $\hat{\pi}_n^k: \mathcal{B}^{(k)} \rightarrow \mathcal{B}^{(n)}$ is the standard projection.

Proposition 2.11. Let $\hat{\rho}^{(n)}$ be an n^{th} order right moving frame. The *normalized invariants*

$$(s, H^{(n)}, I^{(n)}) = \iota(s, x^{(n)}, u^{(n)}) = \rho^{(n)}(\mathfrak{z}^{(n)}) \cdot \mathfrak{z}^{(n)},$$

form a complete set of differential invariants of order $\leq n$.

Example 2.12. In this example we construct a moving frame for the pseudo-group (1.1). The computations for graphs of functions $(x, y, u(x, y))$ appear in [22]. In preparation for the next section we revisit the calculations using the computational variables (s, t) so that $x = x(s, t)$, $y = y(s, t)$ and $u = u(s, t)$. To simplify the computations let

$$y = kt + y_0, \quad (2.15a)$$

where $k > 0$ and y_0 are constants, and assume that

$$x_t = 0. \quad (2.15b)$$

In other words, $x = x(s)$ is a function of the computational variable s . We note that the constraints (2.15) are invariant under the pseudo-group action (1.1). For the y variable, this is straightforward as it is an invariant. The invariance of (2.15b) follows from the chain rule:

$$X_t = f_x x_t = 0 \quad \text{when} \quad x_t = 0.$$

The non-degeneracy condition (2.8) requires the invariant constraint $x_s \neq 0$ to be satisfied.

Up to order 2, the prolonged action is

$$\begin{aligned} S &= s, & T &= t, & Y &= y, & X &= f(x), & U &= \frac{u}{f_x}, \\ X_s &= f_x x_s, & Y_t &= k, & U_s &= \frac{u_s}{f_x} - \frac{u f_{xx} x_s}{f_x^2}, & U_t &= \frac{u_t}{f_x}, \\ X_{ss} &= f_{xx} x_s^2 + f_x x_{ss}, & Y_{tt} &= 0, & U_{tt} &= \frac{u_{tt}}{u}, & U_{st} &= \frac{u_{st}}{f_x} - \frac{u_t f_{xx} x_s}{f_x^2}, \\ U_{ss} &= \frac{u_{ss}}{f_x} + 2 \frac{u f_{xx}^2 x_s^2}{f_x^3} - 2 \frac{u_s f_{xx} x_s}{f_x^2} - \frac{u f_{xxx} x_s^2}{f_x^2} - \frac{u f_{xx} x_{ss}}{f_x^2}. \end{aligned} \quad (2.16)$$

A cross-section to the prolonged action (2.16) and its prolongation is given by

$$\mathcal{K}^{(\infty)} = \{x = 0, u = 1, u_{s^k} = 0, k \geq 1\}. \quad (2.17)$$

Solving the normalization equations

$$X = 0, \quad U = 1, \quad U_{s^k} = 0, \quad k \geq 1,$$

for the pseudo-group parameters f, f_x, f_{xx}, \dots , we obtain the right moving frame

$$f = 0, \quad f_x = u, \quad f_{xx} = \frac{u_s}{x_s}, \quad f_{xxx} = \frac{u_{ss}}{x_s^2} - \frac{u_s x_{ss}}{x_s^3}, \quad \dots$$

In general,

$$f = 0, \quad f_{x^{k+1}} = \left(\frac{D_s}{x_s}\right)^k u, \quad k \geq 0. \quad (2.18)$$

Substituting the pseudo-group normalizations (2.18) into the prolonged action (2.16) yields the normalized differential invariants

$$\begin{aligned} \iota(s) = s, \quad \iota(t) = t, \quad \iota(y) = y, \quad I_1 = \iota(x_s) = u x_s, \quad \iota(y_t) = k, \\ J_{0,1} = \iota(u_t) = \frac{u_t}{u}, \quad I_2 = \iota(x_{ss}) = u_s x_s + u x_{ss}, \quad \iota(y_{tt}) = 0, \\ J_{0,2} = \iota(u_{tt}) = \frac{u_{tt}}{u}, \quad J_{1,1} = \iota(u_{st}) = \frac{u u_{st} - u_t u_s}{u^2}. \end{aligned} \quad (2.19)$$

Remark 2.13. To transition between the expressions obtained in Example 2.12 and those appearing in [22], it suffices to use the chain rule. When (2.15) holds,

$$D_x = \frac{D_s}{x_s}, \quad D_y = \frac{D_t}{k},$$

so that

$$u_x = \frac{u_s}{x_s}, \quad u_y = \frac{u_t}{k}, \quad u_{xx} = \frac{x_s u_{ss} - u_s x_{ss}}{x_s^3}, \quad u_{xy} = \frac{u_{st}}{k x_s}, \quad u_{yy} = \frac{u_{tt}}{k^2}, \quad \dots \quad (2.20)$$

The prolonged action on u_x, u_y, \dots , can then be obtained by substituting (2.16) into (2.20). For example,

$$U_X = \frac{U_s}{X_s} = \left(\frac{u_s}{f_x} - \frac{u f_{xx} x_s}{f_x^2}\right) \frac{1}{f_x x_s} = \frac{u_x}{f_x^2} - \frac{u f_{xx}}{f_x^3}.$$

In the jet variables $z^{(\infty)} = (x, y, u^{(\infty)}) = (x, y, u, u_x, u_y, \dots)$ a cross-section is given by, [22],

$$\overline{\mathcal{K}}^{(\infty)} = \{x = 0, u = 1, u_{x^k} = 0, k \geq 1\}, \quad (2.21)$$

and the corresponding moving frame is

$$f = 0, \quad f_{x^{k+1}} = u_{x^k}, \quad k \geq 0. \quad (2.22)$$

Expressing u_{x^k} in terms of the derivatives x_{s^k}, u_{s^k} using (2.20), one sees that (2.21) and (2.22) are equivalent to (2.17) and (2.18) in the computational variable framework.

In the following, the cross-sections (2.17) and (2.21) (and the corresponding moving frames (2.18) and (2.22)) are said to be *equivalent*.

Not all cross-sections are equivalent. For example, instead of using the cross-section (2.17), it is also possible to choose the (non-minimal) cross-section

$$\tilde{\mathcal{K}}^{(\infty)} = \{x = 0, x_s = 1, x_{s^{k+2}} = 0, k \geq 0\}. \quad (2.23)$$

Since (2.21) is not related to (2.23) by the substitutions (2.20), the cross-section (2.23) is said to be *inequivalent* to (2.21).

Definition 2.14. Let $\mathcal{G}^{(n)}$ be a Lie pseudo-group acting on $J^{(n)}$ and $\mathcal{J}^{(n)}$. A cross-section $\mathcal{K}^{(n)} \subset \mathcal{J}^{(n)}$ is said to be *equivalent* with the cross-section $\bar{\mathcal{K}}^{(n)} \subset J^{(n)}$ if the defining equation (2.14) of $\mathcal{K}^{(n)}$ are obtained from those of $\bar{\mathcal{K}}^{(n)}$ by expressing the submanifold $\text{jet } z^{(n)}$ in terms of $\mathfrak{z}^{(n)}$ using the relations (2.10).

3 Discrete pseudo-groups and moving frames

Let $M^{\times k}$ denote the k -fold Cartesian product of a manifold M . Discrete points in M are labelled using the multi-index notation

$$z_N = (x_N, u_N), \quad N = (n^1, \dots, n^p) \in \mathbb{Z}^p. \quad (3.1)$$

The multi-index notation (3.1) can be related to the continuous theory of Section 2 in the following way. The multi-index $N = (n^1, \dots, n^p) \in \mathbb{Z}^p \subset \mathbb{R}^p$ can be thought as sampling the computational variables $s = (s^1, \dots, s^p) \in \mathbb{R}^p$ on a unit hypercube grid. Thus, the notation $z_N = z(N)$ can be understood as sampling a submanifold $S \subset M$ parameterized by $z(s) = (x(s), u(s))$ at the integer points $N \in \mathbb{Z}^p$.

To mimic the continuous theory of moving frames in the finite difference setting, a discrete counterpart to the submanifold jet space $\mathcal{J}^{(n)}$ is introduced.

Definition 3.1. Let M be a manifold with local coordinate system $z = (x, u)$. The k -fold *joint product* of M is a subset of the k -fold Cartesian product $M^{\times k}$ given by

$$M^{\circ k} = \{(z_{N_1}, \dots, z_{N_k}) \mid z_{N_i} \neq z_{N_j} \text{ for all } i \neq j\} \subset M^{\times k}.$$

Definition 3.2. The n^{th} order *forward discrete jet* at the multi-index N is the point

$$\mathfrak{z}_N^{[n]} = (N, \dots, z_{N+K} \dots), \quad (3.2)$$

where $(\dots, z_{N+K} \dots) \in M^{\circ d_n}$ with

$$d_n = m \binom{p+n}{n}, \quad m = \dim M,$$

and $K = (k^1, \dots, k^p)$ is a non-negative multi-index of order $0 \leq \#K \leq n$.

In dimension 2, when $N = (m, n)$, Figure 2 shows the multi-indices contained in a forward discrete jet of order ≤ 2 . In general, the multi-indices included in $\mathfrak{z}_{m,n}^{[k]}$ are those contained in the interior and boundary of the right isosceles triangle with vertices at (m, n) , $(m+k, n)$ and $(m, n+k)$.

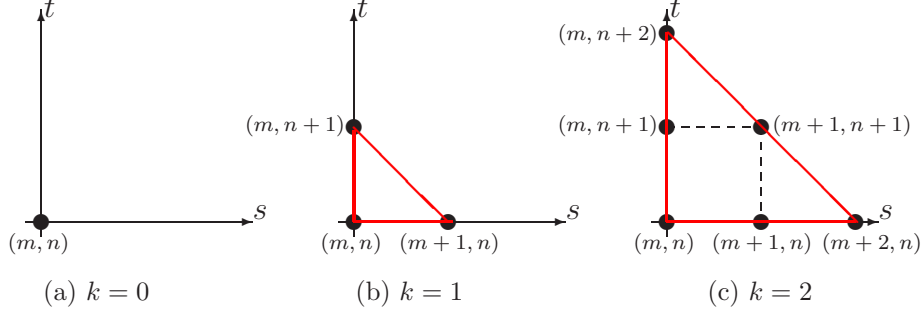


Figure 2: Multi-indices occurring in $\mathfrak{z}_{m,n}^{[k]} \in \mathcal{J}^{[k]}$ for $k = 0, 1, 2$.

Definition 3.3. The n^{th} order forward joint jet space $\mathcal{J}^{[n]}$ is the collection of forward discrete jets (3.2):

$$\mathcal{J}^{[n]} = \bigcup_{N \in \mathbb{Z}^p} \mathfrak{z}_N^{[n]}.$$

For $k > n$, $\pi_n^k: \mathcal{J}^{[k]} \rightarrow \mathcal{J}^{[n]}$ will denote the projection obtained by truncating $\mathfrak{z}_N^{[k]} = (N, \dots, z_{N+K} \dots)$, $0 \leq \#K \leq k$, to

$$\pi_n^k(\mathfrak{z}_N^{[k]}) = \mathfrak{z}_N^{[n]} = (N, \dots, z_{N+K} \dots), \quad 0 \leq \#K \leq n.$$

Let us explain how $\mathcal{J}^{[n]}$ can be understood as a discrete representation of the submanifold jet space $\mathcal{J}^{(n)}$. For this, let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ be the i^{th} element of the standard orthonormal basis of \mathbb{R}^p , and let

$$S_i(N) = N + e_i, \quad i = 1, \dots, p,$$

denote the forward shift operator in the i^{th} component. Then, on a unit hypercube grid in the computational variables, the derivative operators D_{s^i} can be approximated by the forward difference

$$D_{s^i} \sim \Delta_i = S_i - \mathbb{1}, \quad i = 1, \dots, p,$$

where $\mathbb{1}(N) = N$ is the identity map. Then, for a non-negative multi-index $K = (k^1, \dots, k^p)$,

$$z_{s^K}^N = \Delta_1^{k_1} \cdots \Delta_p^{k_p}(z_N) \tag{3.3}$$

is a forward difference approximation of the derivative z_{s^K} at the point $s = N$. Making the change of variables $z_{N+K} \mapsto z_{s^K}^N$, we have that

$$\mathfrak{z}_N^{[n]} \simeq (N, \dots, z_{s^K}^N \dots) = (N, \dots, x_{s^K}^N \dots, u_{s^K}^N \dots), \quad 0 \leq \#K \leq n,$$

is a finite difference approximation of the submanifold jet $(s, x^{(n)}, u^{(n)})$ at the point $s = N$ on a unit hypercube grid. In this sense, $\mathfrak{z}_N^{[n]}$ can be thought as a discrete counterpart to the submanifold jet $\mathfrak{z}^{(n)} = (N, x^{(n)}, u^{(n)})$ in the computational variable formalism, [9].

Remark 3.4. In (3.3) and elsewhere, the usual derivative notation is supplemented by a superscript to denote (forward) discrete derivatives. The superscript indicates where the derivative is evaluated.

Remark 3.5. It is also possible to introduce a backward discrete jet space by introducing the backward differences

$$\nabla_i = \mathbb{1} - S_i^-, \quad \text{where} \quad S_i^-(N) = N - e_i.$$

For numerical purposes, it might be preferable to consider symmetric discrete jets, but to simplify the exposition we restrict ourself to forward differences. All constructions can be adapted to these alternative settings.

Now, assume that the discrete counterpart of the non-degeneracy condition (2.8) holds. Namely,

$$\det(\Delta_j(x_N^i)) \neq 0. \quad (3.4)$$

Then, discrete approximations $u_{x^J}^{\alpha;N}$ of the derivatives $u_{x^J}^\alpha$ can be obtained as follows:

1. compute the expressions (2.10),
2. replace the derivatives D_{s^i} by the difference operators Δ_i .

Since the independent variables x_N^i do not have to form a rectangular grid, the finite difference approximations $u_{x^J}^{\alpha;N}$ will hold on any admissible mesh. Having these expressions will be important as below a Lie pseudo-group will act on $z_N = (x_N, u_N)$ and the expressions for $u_{x^J}^{\alpha;N}$ need to hold on general meshes, [7, 16, 26].

Using the approximations $u_{x^J}^{\alpha;N}$, a finite difference approximation of the jet space $J^{(n)}$ is given by

$$J^{(n)} \sim J^{[n]} = \bigcup_{N \in \mathbb{Z}^p} (x_N, \dots, u_{x^J}^{\alpha;N} \dots), \quad 0 \leq \#J \leq n.$$

Example 3.6. To illustrate the above discussion, we consider the case of two independent variables (x, y) and one dependent variable $u(x, y)$. Introducing the computational variables $(s, t) \in \mathbb{R}^2$ so that $x = x(s, t)$ and $y = y(s, t)$, the implicit total derivative operators (2.9) are

$$D_x = \frac{y_t D_s - y_s D_t}{x_s y_t - y_s x_t}, \quad D_y = \frac{x_s D_t - x_t D_s}{x_s y_t - y_s x_t}, \quad (3.5)$$

with $x_s y_t - y_s x_t \neq 0$. Applying (3.5) to the dependent variable u yields

$$u_x = D_x u = \frac{y_t u_s - y_s u_t}{x_s y_t - y_s x_t} \quad \text{and} \quad u_y = D_y u = \frac{x_s u_t - x_t u_s}{x_s y_t - y_s x_t}. \quad (3.6)$$

Using the multi-index $N = (m, n) \in \mathbb{Z}^2 \subset \mathbb{R}^2$ to sample the computational variables (s, t) at integer values and introducing the shift operators

$$S_1(m, n) = (m + 1, n), \quad S_2(m, n) = (m, n + 1),$$

and the difference operators

$$D_s \sim \Delta = S_1 - \mathbb{1}, \quad D_t \sim \delta = S_2 - \mathbb{1}, \quad (3.7)$$

finite difference approximations of the first order partial derivatives (3.6) are given by

$$u_x^{m,n} = \frac{\delta y_{m,n} \Delta u_{m,n} - \Delta y_{m,n} \delta u_{m,n}}{\Delta x_{m,n} \delta y_{m,n} - \Delta y_{m,n} \delta x_{m,n}}, \quad u_y^{m,n} = \frac{\Delta x_{m,n} \delta u_{m,n} - \delta x_{m,n} \Delta u_{m,n}}{\Delta x_{m,n} \delta y_{m,n} - \Delta y_{m,n} \delta x_{m,n}}, \quad (3.8)$$

provided $\Delta x_{m,n} \delta y_{m,n} - \Delta y_{m,n} \delta x_{m,n} \neq 0$.

The expressions (3.6) and their finite difference approximations (3.8) can be simplified if constraints on the functions $x(s, t)$ and $y(s, t)$ are imposed. For example, in Example 3.36 we will impose the constraints

$$x_t = 0 \quad \text{and} \quad y_{tt} = 0, \quad (3.9)$$

so that $x = x(s)$ and $y = t f(s) + g(s)$, with $f(s) \cdot x'(s) \neq 0$. The operators (3.5) then reduce to

$$D_x = \frac{y_t D_s - y_s D_t}{x_s y_t}, \quad D_y = \frac{D_t}{y_t}, \quad (3.10)$$

and

$$\begin{aligned} u_x &= \frac{y_t u_s - y_s u_t}{x_s y_t}, & u_y &= \frac{u_t}{y_t}, & u_{yy} &= \frac{u_{tt}}{y_t^2}, & u_{xy} &= \frac{y_t u_{st} - y_{st} u_t - y_s u_{tt}}{x_s y_t^2}, \\ u_{xx} &= \frac{1}{x_s} \left[\frac{y_{st} u_s + y_t u_{ss} - y_{ss} u_t - y_s u_{st} - (x_{ss} y_t + x_s y_{st}) u_x}{x_s y_t} - y_s u_{xy} \right], & (3.11) \\ u_{yyy} &= \frac{u_{ttt}}{y_t^3}, & u_{xyy} &= \frac{y_t u_{stt} - 2u_{tt} y_{st} - y_s u_{ttt}}{x_s y_t^3}, & \dots \end{aligned}$$

At the discrete level the differential constraints (3.9) are replaced by

$$\delta x_{m,n} = 0 \quad \text{and} \quad \delta^2 y_{m,n} = \delta y_{m,n+1} - \delta y_{m,n} = y_{m,n+2} - 2y_{m,n+1} + y_{m,n} = 0.$$

This implies that $x_{m,n} = x_m$ is independent of the index n while $y_{m,n} = n f(m) + g(m)$, with $\Delta x_m \delta y_{m,n} = (x_{m+1} - x_m) \cdot f(m) \neq 0$. Making the substitutions (3.7), the expressions (3.11) are approximated by

$$\begin{aligned} u_y^{m,n} &= \frac{\delta u_{m,n}}{\delta y_{m,n}}, & u_x^{m,n} &= \frac{\delta u_{m,n} \Delta u_{m,n} - \Delta y_{m,n} \delta u_{m,n}}{\Delta x_m \delta y_{m,n}}, \\ u_{yy}^{m,n} &= \frac{\delta^2 u_{m,n}}{(\delta y_{m,n})^2}, & u_{xy}^{m,n} &= \frac{\delta y_{m,n} \Delta \delta u_{m,n} - \Delta \delta y_{m,n} \Delta u_{m,n} - \Delta y_{m,n} \delta^2 u_{m,n}}{\Delta x_m (\delta y_{m,n})^2}, \\ u_{yyy}^{m,n} &= \delta^3 u_{m,n} (\delta y_{m,n})^3, & u_{xyy}^{m,n} &= \frac{\delta y_{m,n} \Delta \delta^2 u_{m,n} - 2\delta^2 u_{m,n} \Delta \delta y_{m,n} - \Delta y_{m,n} \delta^3 u_{m,n}}{\Delta x_m (\delta y_{m,n})^3}. \end{aligned} \quad (3.12)$$

We are now interested in the induced action of a Lie pseudo-group on discrete points.

Definition 3.7. Given a Lie pseudo-group \mathcal{G} acting on M , the *pseudo-group product action* on the k -fold Cartesian product $M^{\times k}$ is

$$(g \cdot z_{N_1}, \dots, g \cdot z_{N_k}), \quad g \in \mathcal{G}, \quad (3.13)$$

provided the points $z_{N_1}, \dots, z_{N_k} \in \text{dom } g$.

Remark 3.8. The nature of the product action (3.13) depends on the type of the Lie pseudo-group \mathcal{G} . If the Lie pseudo-group \mathcal{G} is of infinite type, its k -fold product action is no longer a Lie pseudo-group as it is not possible to encapsulate into a system of differential equations the requirement that the same diffeomorphism should act on different points. In this case, the product action only satisfies the defining properties of a pseudo-group. On the other hand, the k -fold product action of a Lie pseudo-group

of finite type, i.e. a local Lie group action, remains a Lie pseudo-group of finite type. Another important distinction between pseudo-groups of finite and infinite types occurs when more copies of the manifold M are appended to the Cartesian product $M^{\times k}$. For pseudo-groups of infinite type, when a new copy of M is added, new pseudo-group parameters will occur in the product action while this is not the case for pseudo-groups of finite type.

As shown in Example 1.1, no joint invariant of the product action (1.5) can approximate the differential invariant (1.2). This is not peculiar to this pseudo-group and another instance is given in Example 3.21. To address this lack of joint invariants, we propose to discretize the product pseudo-group action, replacing derivatives by finite difference approximations. Before stating the general theory, our proposed idea is applied to the product pseudo-group (1.5).

Example 3.9. On the rectangular grid

$$\delta x_{m,n} = x_{m,n+1} - x_{m,n} = 0, \quad y_n = k n + y_0,$$

a suitable discretization of the product pseudo-group action (1.5) is obtained by approximating the first order derivative $f_x(x_m)$ by the forward difference

$$f_x(x_m) \sim f_x^m = \frac{\Delta f_m}{\Delta x_m} = \frac{f_{m+1} - f_m}{x_{m+1} - x_m} \quad \text{where} \quad f_{m+j} = f(x_{m+j}), \quad (3.14)$$

to give the discretized action

$$\mathcal{G}_d: \quad X_m = f_m, \quad Y_n = y_n, \quad U_{m,n} = \frac{u_{m,n}}{f_x^m}. \quad (3.15)$$

The subscript d is added to \mathcal{G} to indicate that the pseudo-group action has been discretized. For (3.15) to be a legitimate discretization it must satisfy the properties of an action. These are readily seen to be satisfied except maybe for closure under composition. To this end, let

$$\tilde{X}_m = \tilde{f}_m = \tilde{f}(X_m), \quad \tilde{Y}_n = Y_n = y_n, \quad \tilde{U}_{m,n} = \frac{U_{m,n}}{\tilde{f}_X^m},$$

be a second discretized transformation. Then $\tilde{X}_m = \tilde{f} \circ f(x_m)$, and

$$\begin{aligned} \tilde{U}_{m,n} &= U_{m,n} \cdot \frac{\Delta X_m}{\Delta[\tilde{f}(X_m)]} = u_{m,n} \cdot \frac{\Delta x_m}{\Delta[f(x_m)]} \cdot \frac{\Delta[f(x_m)]}{\Delta[\tilde{f} \circ f(x_m)]} \\ &= u_{m,n} \cdot \frac{\Delta x_m}{\Delta[\tilde{f} \circ f(x_m)]} = \frac{u_{m,n}}{(f \circ f)_x^m}, \end{aligned}$$

showing that (3.15) is closed under composition.

The approximation (3.14) is not unique. Any other discretization preserving the group action properties is acceptable. For example, the approximation (3.14) could be replaced by the backward difference

$$f_x(x_m) \sim \frac{f_m - f_{m-1}}{x_m - x_{m-1}}.$$

On the other hand, (3.15) is not closed under composition if the centred approximation

$$f_x(x_m) \sim \frac{1}{2} \left[\frac{\Delta f_m}{\Delta x_m} + \frac{\Delta f_{m-1}}{\Delta x_{m-1}} \right]$$

is considered.

At the infinitesimal level, the discretized action (3.15) is generated by the vector field

$$\mathbf{v}_a = a_m \frac{\partial}{\partial x_m} - u_{m,n} a_x^m \frac{\partial}{\partial u_{m,n}}, \quad (3.16)$$

where

$$a_m = a(x_m) \quad \text{and} \quad a_x^m = \frac{a_{m+1} - a_m}{x_{m+1} - x_m}.$$

To compute the Lie algebra structure of (3.16), one would use the standard prolongation, [16],

$$\mathbf{pr} \mathbf{v}_a = \sum_m a_m \frac{\partial}{\partial x_m} - \sum_{m,n} u_{m,n} a_x^m \frac{\partial}{\partial u_{m,n}}$$

and define the Lie bracket $[\mathbf{v}_a, \mathbf{v}_b]$ of two infinitesimal generators to be the vector field satisfying

$$\mathbf{pr} [\mathbf{v}_a, \mathbf{v}_b] := \mathbf{pr} \mathbf{v}_a \circ \mathbf{pr} \mathbf{v}_b - \mathbf{pr} \mathbf{v}_b \circ \mathbf{pr} \mathbf{v}_a.$$

For two infinitesimal generators of the form (3.16), we obtain the expected commutation relation

$$[\mathbf{v}_a, \mathbf{v}_b] = \mathbf{v}_{ab' - ba'}.$$

Remark 3.10. By introducing the approximation (3.14), the discretized action (3.15) is no longer local as the approximation (3.14) introduces the extra independent variable x_{m+1} into the action at $(x_m, y_n, u_{m,n})$. This type of non-local discrete transformations is reminiscent of transformations obtained when considering discrete generalized symmetries, [16]. Similar pseudo-group discretization also recently appeared in a discrete version of Noether's Second Theorem, [11].

Given an admissible discrete pseudo-group action, it is possible to implement the moving frame method in a fashion similar to the continuous setting. In the continuous theory, the jets of functions occurring in the prolonged action play the role of the pseudo-group parameters. In the discrete case, the functions evaluated at distinct points will play the role of the pseudo-group parameters.

Example 3.11. At the point $(x_m, y_n, u_{m,n})$, the discrete pseudo-group action

$$X_m = f_m = f(x_m), \quad Y_n = y_n, \quad U_{m,n} = u_{m,n} \cdot \frac{x_{m+1} - x_m}{f_{m+1} - f_m}$$

involves the pseudo-group parameters $g_{m,n} = (f_m, f_{m+1})$.

In the following, the pseudo-group parameters occurring in the discrete pseudo-group action at z_N is denoted g_N . To approximate the n^{th} order lifted bundle (2.11) we introduce the n^{th} order *forward joint lifted bundle* $\mathcal{B}^{[n]}$ parameterized by $(\mathfrak{z}_N^{[n]}, g_N^{[n]})$, where

$$g_N^{[n]} = (\dots g_{N+K} \dots), \quad 0 \leq \#K \leq n.$$

The fibre of the n^{th} order forward joint lifted bundle $\mathcal{B}^{[n]}$ at $\mathfrak{z}_N^{[n]}$ is denoted $\mathcal{G}_N^{[n]}$. The discretized pseudo-group \mathcal{G}_d acts on $\mathcal{B}^{[n]}$ by right multiplication

$$R_{h_N}(\mathfrak{z}_N^{[n]}, g_N^{[n]}) = (h_N^{[n]} \cdot \mathfrak{z}_N^{[n]}, (g \cdot h^{-1})^{[n]}|_{h_N^{[n]} \cdot \mathfrak{z}_N^{[n]}}).$$

Definition 3.12. Let \mathcal{G}_d be a discretized Lie pseudo-group acting on the n^{th} order joint lifted bundle $\mathcal{B}^{[n]}$. An order n (right) *joint moving frame* is a \mathcal{G}_d -equivariant section of the order n joint lifted bundle $\mathcal{B}^{[n]}$:

$$\widehat{\rho}^{[n]}(\mathfrak{z}_N^{[n]}) = (\mathfrak{z}_N^{[n]}, \rho^{[n]}(\mathfrak{z}_N^{[n]})).$$

Right equivariance means that

$$R_{g_N} \widehat{\rho}^{[n]}(\mathfrak{z}_N^{[n]}) = \widehat{\rho}^{[n]}(g_N^{[n]} \cdot \mathfrak{z}_N^{[n]}).$$

As in the continuous setting, a moving frame exists on (an open set of) the n^{th} order joint bundle $\mathcal{J}^{[n]}$ if the action is free and regular.

Definition 3.13. A discretized Lie pseudo-group \mathcal{G}_d acts freely at $\mathfrak{z}_N^{[n]}$ if the isotropy group

$$\mathcal{G}_{N; \mathfrak{z}_N^{[n]}}^{[n]} = \{g_N^{[n]} : g_N^{[n]} \cdot \mathfrak{z}_N^{[n]} = \mathfrak{z}_N^{[n]}\} = \{\mathbb{1}_N^{[n]}\}, \quad (3.17)$$

where $\mathbb{1}_N^{[n]}$ is the discrete identity transformation at $\mathfrak{z}_N^{[n]}$.

Example 3.14. For the discretized pseudo-group (3.15), the isotropy condition $\mathfrak{g}_{m,n}^{[0]} \cdot \mathfrak{z}_{m,n}^{[0]} = \mathfrak{z}_{m,n}^{[0]}$ is

$$x_m = f_m, \quad y_n = y_n, \quad u_{m,n} = \frac{u_{m,n}}{f_x^m}$$

which requires

$$f_m = x_m, \quad f_{m+1} = x_{m+1}.$$

In general, the isotropy condition $\mathfrak{g}_{m,n}^{[k]} \cdot \mathfrak{z}_{m,n}^{[k]} = \mathfrak{z}_{m,n}^{[k]}$ yields

$$f_{m+\ell} = x_{m+\ell}, \quad \ell = 0, \dots, k+1.$$

Provided the discrete product pseudo-group action is free and regular, a joint moving frame is constructed through a normalization procedure similar to the continuous case. Let $\mathcal{K}^{[n]} = \{z_{i_1} = c_1, \dots, z_{i_{r_n}} = c_{r_n}\} \subset \mathcal{J}^{[n]}$ be a coordinate cross-section, then the corresponding joint moving frame $\widehat{\rho}^{[n]}$ is obtained by solving the normalization equations

$$Z_{i_1}(\mathfrak{z}_N^{[n]}, g_N^{[n]}) = c_1, \quad \dots \quad Z_{i_{r_n}}(\mathfrak{z}_N^{[n]}, g_N^{[n]}) = c_{r_n}$$

for the pseudo-group parameters $g_N^{[n]} = \rho^{[n]}(\mathfrak{z}_N^{[n]})$. For $k > n$, the cross-sections are required to be compatible, that is $\pi_n^k(\mathcal{K}^{[k]}) = \mathcal{K}^{[n]}$. The corresponding moving frames are then compatible $\widehat{\pi}_n^k \circ \widehat{\rho}^{[n]} = \widehat{\rho}^{[n]} \circ \pi_n^k$. Here $\widehat{\pi}_n^k : \mathcal{B}^{[k]} \rightarrow \mathcal{B}^{[n]}$ is the standard projection obtained by truncation. A discrete analogue of Theorem 2.9 also holds.

Theorem 3.15. Let \mathcal{G}_d be the discretization of a Lie pseudo-group and assume it acts freely at $\mathfrak{z}_N^{[n]}$ for any $N \in \mathbb{Z}^p$. Then for $k > n$, \mathcal{G}_d acts freely at $\mathfrak{z}_N^{[k]}$.

Remark 3.16. Before proving Theorem 3.15 in general, it is instructive to consider a low dimensional example. In two dimensions, assume the discretized action is free at $\mathfrak{z}_{m,n}^{[2]}$. Our goal is to show that it remains free at $\mathfrak{z}_{m,n}^{[3]}$. In Figure 3a, the multi-indices contained in $\mathfrak{z}_{m,n}^{[3]}$ are displayed. Figures 3b–3d show that sitting inside $\mathfrak{z}_{m,n}^{[3]}$ are the order 2 discrete jets

$$\mathfrak{z}_{m,n}^{[2]}, \quad \mathfrak{z}_{m+1,n}^{[2]}, \quad \mathfrak{z}_{m,n+1}^{[2]}.$$

Since these three order 2 jets cover $\mathfrak{z}_{m,n}^{[3]}$,

$$\mathfrak{z}_{m,n}^{[3]} \simeq (\mathfrak{z}_{m,n}^{[2]}, \mathfrak{z}_{m+1,n}^{[2]}, \mathfrak{z}_{m,n+1}^{[2]}). \quad (3.18a)$$

Similarly, at the pseudo-group level

$$g_{m,n}^{[3]} \simeq (g_{m,n}^{[2]}, g_{m+1,n}^{[2]}, g_{m,n+1}^{[2]}). \quad (3.18b)$$

Next, a pseudo-group transformation $g_{m,n}^{[3]} \in \mathcal{G}_{m,n}^{[3]}$ keeps $\mathfrak{z}_{m,n}^{[3]}$ fixed if and only if it keeps $\mathfrak{z}_{m,n}^{[2]}, \mathfrak{z}_{m+1,n}^{[2]}, \mathfrak{z}_{m,n+1}^{[2]}$ fixed simultaneously. That is

$$\mathcal{G}_{m,n;\mathfrak{z}_{m,n}^{[3]}}^{[3]} = \mathcal{G}_{m,n;\mathfrak{z}_{m,n}^{[2]}}^{[3]} \cap \mathcal{G}_{m,n;\mathfrak{z}_{m+1,n}^{[2]}}^{[3]} \cap \mathcal{G}_{m,n;\mathfrak{z}_{m,n+1}^{[2]}}^{[3]}, \quad (3.19)$$

where

$$\begin{aligned} \mathcal{G}_{m,n;\mathfrak{z}_{m,n}^{[2]}}^{[3]} &= \{g_{m,n}^{[3]} : g_{m,n}^{[3]} \cdot \mathfrak{z}_{m,n}^{[2]} = g_{m,n}^{[2]} \cdot \mathfrak{z}_{m,n}^{[2]} = \mathfrak{z}_{m,n}^{[2]}\}, \\ \mathcal{G}_{m,n;\mathfrak{z}_{m+1,n}^{[2]}}^{[3]} &= \{g_{m,n}^{[3]} : g_{m,n}^{[3]} \cdot \mathfrak{z}_{m+1,n}^{[2]} = g_{m+1,n}^{[2]} \cdot \mathfrak{z}_{m+1,n}^{[2]} = \mathfrak{z}_{m+1,n}^{[2]}\}, \\ \mathcal{G}_{m,n;\mathfrak{z}_{m,n+1}^{[2]}}^{[3]} &= \{g_{m,n}^{[3]} : g_{m,n}^{[3]} \cdot \mathfrak{z}_{m,n+1}^{[2]} = g_{m,n+1}^{[2]} \cdot \mathfrak{z}_{m,n+1}^{[2]} = \mathfrak{z}_{m,n+1}^{[2]}\}. \end{aligned}$$

By assumption

$$\mathcal{G}_{m,n;\mathfrak{z}_{m,n}^{[2]}}^{[2]} = \{\mathbb{1}_{m,n}^{[2]}\}, \quad \mathcal{G}_{m+1,n;\mathfrak{z}_{m+1,n}^{[2]}}^{[2]} = \{\mathbb{1}_{m+1,n}^{[2]}\}, \quad \mathcal{G}_{m,n+1;\mathfrak{z}_{m,n+1}^{[2]}}^{[2]} = \{\mathbb{1}_{m,n+1}^{[2]}\},$$

and it follows that

$$\begin{aligned} \mathcal{G}_{m,n;\mathfrak{z}_{m,n}^{[3]}}^{[3]} &\simeq \{(\mathbb{1}_{m,n}^{[2]}, *, *)\}, \quad \mathcal{G}_{m,n;\mathfrak{z}_{m+1,n}^{[2]}}^{[3]} \simeq \{(*, \mathbb{1}_{m+1,n}^{[2]}, *)\}, \\ \mathcal{G}_{m,n;\mathfrak{z}_{m,n+1}^{[2]}}^{[3]} &\simeq \{(*, *, \mathbb{1}_{m,n+1}^{[2]})\}, \end{aligned} \quad (3.20)$$

under the isomorphism (3.18). The exact expressions for $*$ will depend on the particular pseudo-group action. Combining (3.19) and (3.20), we conclude that

$$\mathcal{G}_{m,n;\mathfrak{z}_{m,n}^{[3]}}^{[3]} \simeq \{(\mathbb{1}_{m,n}^{[2]}, \mathbb{1}_{m+1,n}^{[2]}, \mathbb{1}_{m,n+1}^{[2]})\} \simeq \{\mathbb{1}_{m,n}^{[3]}\}.$$

Proof. To prove Theorem 3.15, it suffices to consider the case when $k = n + 1$ and proceed as in the 2-dimensional example above. First,

$$\mathfrak{z}_N^{[n+1]} \simeq (\dots \mathfrak{z}_{N+e_i}^{[n]} \dots), \quad i = 1, \dots, p.$$

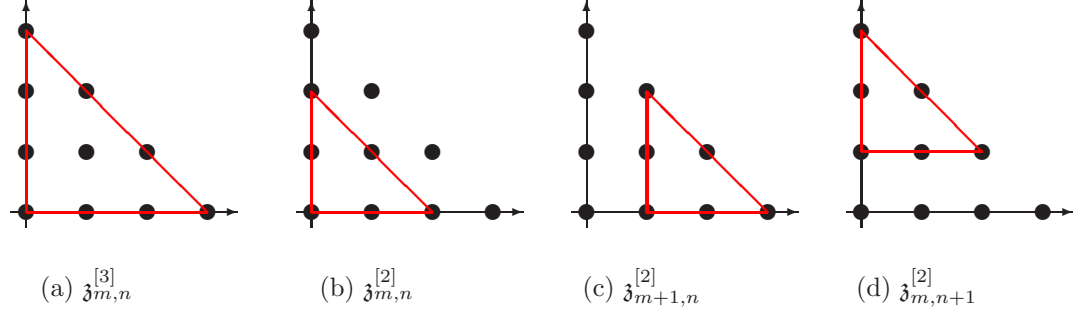


Figure 3: Forward discrete jets of order 2 contained in $\mathfrak{z}_{m,n}^{[3]}$.

Next, since by assumption

$$\mathcal{G}_{N+e_i; \mathfrak{z}_{N+e_i}^{[n]}}^{[n]} = \{\mathbb{1}_{N+e_i}^{[n]}\}, \quad i = 1, \dots, p,$$

one has that

$$\begin{aligned} \mathcal{G}_{N; \mathfrak{z}_N^{[n]}}^{[n+1]} &= \{g_N^{[n+1]} : g_N^{[n+1]} \cdot \mathfrak{z}_{N+e_i}^{[n]} = g_{N+e_i}^{[n]} \cdot \mathfrak{z}_{N+e_i}^{[n]} = \mathfrak{z}_{N+e_i}^{[n]}\} \\ &\simeq \{(\dots, *, \dots, \mathbb{1}_{N+e_i}^{[n]}, \dots, *, \dots)\}, \end{aligned}$$

and

$$\mathcal{G}_{N; \mathfrak{z}_N^{[n+1]}}^{[n+1]} = \bigcap_{i=1}^p \mathcal{G}_{N; \mathfrak{z}_{N+e_i}^{[n]}}^{[n+1]} = \{\mathbb{1}_N^{[n+1]}\}.$$

□

Definition 3.17. A function $I(\mathfrak{z}_N^{[n]}): \mathcal{J}^{[n]} \rightarrow \mathbb{R}$ is a *joint invariant* if

$$I(g_N^{[n]} \cdot \mathfrak{z}_N^{[n]}) = I(\mathfrak{z}_N^{[n]}), \quad g_N^{[n]} \in \mathcal{G}_N^{[n]},$$

whenever the discrete product action is defined.

Definition 3.18. Let $\hat{\rho}^{[n]}(\mathfrak{z}_N^{[n]})$ be an order n joint moving frame. The *invariantization* of a function $F(\mathfrak{z}_N^{[n]})$ is the joint invariant

$$\iota(F)(\mathfrak{z}_N^{[n]}) = F(\rho^{[n]}(\mathfrak{z}_N^{[n]}) \cdot \mathfrak{z}_N^{[n]}). \quad (3.21)$$

Of particular interest to us is the invariantization of the discrete derivatives $u_{x^J}^{\alpha; N}$:

$$I_J^{\alpha; N} = \iota(u_{x^J}^{\alpha; N}), \quad \alpha = 1, \dots, q, \quad \#J \geq 0. \quad (3.22)$$

We say that the cross-section $\mathcal{K}^{[n]}$ used to construct a joint moving frame $\hat{\rho}^{[n]}$ is *consistent* with the cross-section $\mathcal{K}^{(n)}$ used to construct a (continuous) moving frame $\hat{\rho}^{(n)}$ if, in the continuous limit, $\mathcal{K}^{[n]}$ converges to $\mathcal{K}^{(n)}$. For consistent cross-sections, since the discretized pseudo-group action \mathcal{G}_d converges to the Lie pseudo-group \mathcal{G} in the continuous limit, the discrete invariants (3.22) will converge to the differential invariants $I_J^\alpha = \iota(u_{x^J}^\alpha)$:

$$\iota(u_{x^J}^{\alpha; N}) = I_J^{\alpha; N} \rightarrow I_J^\alpha = \iota(u_{x^J}^\alpha).$$

Example 3.19. In this example, a joint moving frame for the discretized pseudo-group action (3.15) is constructed. First, a cross-section is given by

$$\mathcal{K}^{[\infty]} = \{x_m = 0, u_{m+k,n} = 1, k \in \mathbb{N}\}. \quad (3.23)$$

Written differently, the cross-section is equivalent to

$$x_m = 0, \quad u_{m,n} = 1, \quad \Delta^k(u_{m,n}) = 0, \quad k = 1, 2, \dots, \quad (3.24)$$

which is an approximation of (2.17) on a unit square mesh in the computational variables (s, t) . Hence, (3.23) is consistent with the cross-section (2.17) used in the continuous setting. Solving the normalization equations

$$0 = X_m = f_m, \quad 1 = U_{m+k,n} = \frac{u_{m+k,n}}{f_x^{m+k}} = u_{m+k,n} \cdot \frac{\Delta x_{m+k}}{\Delta f_{m+k}},$$

for the pseudo-group parameters f_{m+k} , $k \geq 0$, produces the (forward) joint moving frame

$$f_m = 0, \quad f_{m+k} = \sum_{l=1}^k u_{m+l-1,n} \Delta x_{m+l-1}, \quad k = 1, 2, 3, \dots \quad (3.25)$$

Applying the invariantization map (3.21) to the discrete variables x_{m+k} , y_{n+l} , $u_{m+k,n+l}$, we obtain the normalized joint invariants

$$\begin{aligned} \iota(x_{m+k}) &= (\rho^{[\infty]})^* f_{m+k} = \begin{cases} 0 & k = 0, \\ \sum_{l=1}^k u_{m+l-1,n} \Delta x_{m+l-1} & k = 1, 2, 3, \dots, \end{cases} \\ \iota(y_{n+l}) &= y_{n+l}, \quad \iota(u_{m+k,n+l}) = \frac{u_{m+k,n+l}}{(\rho^{[\infty]})^* f_x^{m+k}} = \frac{u_{m+k,n+l}}{u_{m+k,n}}, \quad k, l = 0, 1, 2, \dots \end{aligned} \quad (3.26)$$

Alternatively, invariantizing the forward differences in x_m , y_n , $u_{m,n}$ yields the joint invariants

$$\begin{aligned} \iota(y_n) &= y_n, \quad I_1^d = \iota(\Delta x_m) = u_{m,n} \Delta x_m, \quad \iota(\delta y_n) = k, \\ J_{0,1}^d &= \iota(\delta u_{m,n}) = \frac{\delta u_{m,n}}{u_{m,n}}, \quad I_2^d = \iota(\Delta^2 x_m) = \Delta u_{m,n} \Delta x_m + u_{m,n} \Delta^2 x_m, \quad \iota(\delta^2 y_n) = 0, \\ J_{0,2}^d &= \iota(\delta^2 u_{m,n}) = \frac{\delta^2 u_{m,n}}{u_{m,n}}, \quad J_{1,1}^d = \iota(\Delta \delta u_{m,n}) = \frac{u_{m,n} \Delta \delta u_{m,n} - \delta u_{m,n} \Delta u_{m,n}}{u_{m+1,n} u_{m,n}}. \end{aligned}$$

These invariants are finite difference approximations of the normalized differential invariants (2.19) on a unit square grid in the computational variables. Another possibility is to invariantize the discrete derivatives

$$u_y^{m,n} = \frac{\delta u_{m,n}}{\delta y_n}, \quad u_{xy}^{m,n} = \frac{\Delta \delta u_{m,n}}{\delta y_n \Delta x_m}, \quad u_{yy}^{m,n} = \frac{\delta^2 u_{m,n}}{(\delta y_n)^2},$$

to obtain the joint invariants

$$\begin{aligned} I_{0,1}^d &= \iota(u_y^{m,n}) = \frac{u_y^{m,n}}{u_{m,n}}, \quad I_{0,2}^d = \iota(u_{yy}^{m,n}) = \frac{u_{yy}^{m,n}}{u_{m,n}}, \\ I_{1,1}^d &= \iota(u_{xy}^{m,n}) = \frac{u_{m+1,n+1} u_{m,n} - u_{m+1,n} u_{m,n+1}}{u_{m,n}^2 u_{m+1,n} \Delta x_m \delta y_n} = \frac{u_{m,n} u_{xy}^{m,n} - u_x^{m,n} u_y^{m,n}}{u_{m,n}^2 u_{m+1,n}}. \end{aligned} \quad (3.27)$$

In the continuous limit, the invariants (3.27) converge to the normalized differential invariants obtained in [22].

Our main illustrative pseudo-group (1.1) was chosen for its simplicity. This pseudo-group can be embedded into the larger pseudo-group

$$X = f(x), \quad Y = g(y), \quad U = \frac{u}{f_x g_y}, \quad f, g \in \mathcal{D}(\mathbb{R}), \quad (3.28)$$

which, as observed in the introduction, is the full symmetry group of the differential equation (1.3). In the following example, joint invariants of the discretized action (3.28) are computed. The results of these computations will be used in Sections 4 and 5 to construct a fully invariant numerical scheme of equation (1.3) and perform numerical tests.

Example 3.20. The construction of a joint moving frame for the pseudo-group (3.28) is similar to the previous example. Though, one important difference between the two examples is that it is no longer possible to work under the assumption that the step size $\delta y_n = k$ is constant as this is not an invariant constraint of the larger pseudo-group (3.28). The most one can impose is that the mesh be rectangular

$$\delta x_{m,n} = x_{m,n+1} - x_{m,n} = 0, \quad \Delta y_{m,n} = y_{m+1,n} - y_{m,n} = 0, \quad (3.29)$$

so that $x_{m,n} = x_m$ and $y_{m,n} = y_n$. At the discrete level, the pseudo-group action (3.28) can be approximated by the forward discrete action

$$X_m = f_m = f(x_m), \quad Y_n = g_n = g(y_n), \quad U_{m,n} = \frac{u_{m,n}}{f_x^m g_y^n}, \quad (3.30)$$

where

$$f_x^m = \frac{\Delta f_m}{\Delta x_m} = \frac{f_{m+1} - f_m}{x_{m+1} - x_m}, \quad g_y^n = \frac{\delta g_n}{\delta y_n} = \frac{g_{n+1} - g_n}{y_{n+1} - y_n}.$$

Summarizing the moving frame construction, a cross-section is given by

$$\begin{aligned} \mathcal{K}^{[\infty]}: \quad x_m = 0, \quad y_n = 0, \quad \Delta x_m \delta \Delta^2 u_{m,n} - \Delta^2 x_m \delta \Delta u_{m,n} &= (\Delta x_m)^3 \delta y_n, \\ u_{m+k,n} = u_{m,n+k} &= 1, \quad k \geq 0, \end{aligned}$$

and the corresponding joint moving frame is

$$\begin{aligned} f_m = 0, \quad f_{m+k} &= \frac{\delta y_n}{g_{n+1}} \sum_{l=0}^{k-1} u_{m+l,n} \Delta x_{m+l}, \\ g_n = 0, \quad g_{n+k} &= \frac{g_{n+1}}{u_{m,n} \delta y_n} \sum_{l=0}^{k-1} u_{m,n+l} \delta y_{n+l}, \end{aligned} \quad (3.31a)$$

where $k \geq 1$, and

$$g_{n+1} = \frac{u_{m,n+1} u_{m,n} (\Delta x_m)^2 (\delta y_n)^2}{u_{m+1,n} \Delta x_{m+1} \Delta \left[\frac{1}{u_{m,n} \Delta x_m} \Delta \left(\frac{u_{m,n+1}}{u_{m,n}} \right) \right]}. \quad (3.31b)$$

The applications of the invariantization map (3.21) to the discrete variables x_{m+k} , y_{n+l} , $u_{m+k,n+l}$ yields the normalized joint invariants

$$\begin{aligned} \iota(x_{m+k}) &= (\rho^{[\infty]})^* f_{m+k} = \begin{cases} 0 & k = 0, \\ \frac{\delta y_n}{g_{n+1}} \sum_{l=0}^{k-1} u_{m+l,n} \Delta x_{m+l} & k = 1, 2, 3, \dots, \end{cases} \\ \iota(y_{n+k}) &= (\rho^{[\infty]})^* g_{n+k} = \begin{cases} 0 & k = 0, \\ \frac{g_{n+1}}{u_{m,n} \delta y_n} \sum_{l=0}^{k-1} u_{m,n+l} \delta y_{n+l} & k = 1, 2, 3, 4, \dots, \end{cases} \\ \iota(u_{m+k,n+l}) &= \frac{u_{m+k,n+l}}{(\rho^{[\infty]})^* f_x^{m+k} (\rho^{[\infty]})^* g_y^{n+k}} = \frac{u_{m+k,n+l} u_{m,n}}{u_{m+k,n} u_{m,n+k}}, \quad k, l = 0, 1, 2, \dots \end{aligned}$$

For later use, the invariantization of

$$u_{xy}^{m,n} = \frac{\delta \Delta u_{m,n}}{\Delta x_m \delta y_n}$$

gives the joint invariant

$$I_{1,1}^d = \iota(u_{xy}^{m,n}) = \frac{u_{m+1,n+1} u_{m,n} - u_{m+1,n} u_{m,n+1}}{u_{m,n} u_{m+1,n} u_{m,n+1} \Delta x_m \delta y_n} = \frac{u_{m,n} u_{xy}^{m,n} - u_x^{m,n} u_y^{m,n}}{u_{m,n} u_{m+1,n} u_{m,n+1}}. \quad (3.32)$$

Example 3.21. The Lie pseudo-group

$$X = f(x), \quad Y = e(x, y) = y f_x + g(x), \quad U = u + \frac{e_x}{f_x} = u + \frac{y f_{xx} + g_x}{f_x}, \quad (3.33)$$

will serve as our last example. This pseudo-group was used by Vessiot in his work on automorphic systems, [29]. It is also one of the pseudo-groups used in [22] to illustrate the method of equivariant moving frames.

By a similar argument to Example 1.1, on a generic mesh $(x_{m,n}, y_{m,n})$, the product pseudo-group action

$$\begin{aligned} X_{m,n} &= f(x_{m,n}), \quad Y = e(x_{m,n}, y_{m,n}) = y_{m,n} f_x(x_{m,n}) + g(x_{m,n}), \\ U_{m,n} &= u_{m,n} + \frac{e_x(x_{m,n}, y_{m,n})}{f_x(x_{m,n})} \end{aligned} \quad (3.34)$$

has no joint invariant since $f(x_{m,n})$, $e(x_{m,n}, y_{m,n})$, and $e_x(x_{m,n}, y_{m,n})/f_x(x_{m,n})$ are generically independent. To reduce the number of pseudo-group parameters as much as possible, the invariant constraints

$$\delta x_{m,n} = x_{m,n+1} - x_{m,n} = 0, \quad \delta^2 y_{m,n} = y_{m,n+2} - 2y_{m,n+1} + y_{m,n} = 0 \quad (3.35)$$

are imposed. Note that it is not possible to invariantly assume $\Delta y_{m,n} = y_{m+1,n} - y_{m,n} = 0$. Hence, rectangular meshes are not invariant for this pseudo-group. Provided $\delta y_{m,n} \neq 0$, which is an invariant constraint of (3.34) when (3.35) is satisfied, the product pseudo-group action

$$X_m = f(x_m), \quad Y = e(x_m, y_{m,n}) = y_{m,n} f_x(x_m) + g(x_m), \quad U_{m,n} = u_{m,n} + \frac{e_x(x_m, y_{m,n})}{f_x(x_m)}$$

admits the joint invariants

$$\frac{y_{m,n+k} - y_{m,n}}{y_{m,n+1} - y_{m,n}}, \quad u_{m,n+k} - u_{m,n} - \left(\frac{y_{m,n+k} - y_{m,n}}{y_{m,n+1} - y_{m,n}} \right) (u_{m,n+1} - u_{m,n}), \quad k \in \mathbb{Z}.$$

By the same dilation argument as in Example 1.1, it is possible to conclude that these joint invariants cannot approximate all the differential invariants obtained in [22]. To construct further joint invariants the product action (3.34) is discretized. An admissible discretization is given by

$$\begin{aligned} X_m &= f_m = f(x_m), & Y_{m,n} &= e_{m,n} = y_{m,n} f_x^m + g_m, \\ U_{m,n} &= u_{m,n} + \frac{e_x^{m,n}}{f_x^m} = u_{m,n} + \frac{\Delta e_{m,n}}{\Delta f_m} - \frac{\Delta y_{m,n}}{\Delta x_m} = u_{m,n} + \frac{y_{m+1,n}}{\Delta f_m} \Delta \left(\frac{\Delta f_m}{\Delta x_m} \right) + \frac{\Delta g_m}{\Delta f_m}, \end{aligned} \quad (3.36)$$

where

$$g_m = g(x_m), \quad f_x^m = \frac{\Delta f_m}{\Delta x_m} \quad \text{and} \quad e_x^{m,n} = \frac{\Delta e_{m,n}}{\Delta x_m} - \frac{\Delta y_{m,n} \delta e_{m,n}}{\Delta x_m \delta y_{m,n}}.$$

To verify closure of (3.36) under composition, let

$$\tilde{X}_m = \tilde{f}_m, \quad \tilde{Y}_{m,n} = \tilde{e}_{m,n} = \frac{\Delta \tilde{f}_m}{\Delta X_m} Y_{m,n} + \tilde{g}_m, \quad \tilde{U}_{m,n} = U_{m,n} + \frac{\Delta \tilde{e}_{mn}}{\Delta \tilde{f}_m} - \frac{\Delta Y_{mn}}{\Delta X_m}.$$

Thus, in the x variable $\tilde{X}_m = \tilde{f}_m = \tilde{f}(X_m) = \bar{f}_m \circ f_m$, while in the y variable

$$\tilde{Y}_{mn} = \tilde{e}_{m,n} = \frac{\Delta \tilde{f}_m}{\Delta X_m} \left(\frac{\Delta f_m}{\Delta x_m} y_{m,n} + g_m \right) + \tilde{g}_m = \frac{\Delta \tilde{f}_m}{\Delta x_m} y_{m,n} + G_m,$$

with

$$G_m = \frac{\Delta \tilde{f}_m}{\Delta f_m} g_m + \tilde{g}_m.$$

Finally, in the u variable

$$\begin{aligned} \tilde{U}_{mn} &= u_{mn} + \frac{\Delta e_{mn}}{\Delta f_m} - \frac{\Delta y_{mn}}{\Delta x_m} + \frac{\Delta \tilde{e}_{mn}}{\Delta \tilde{f}_m} - \frac{\Delta Y_{mn}}{\Delta X_m} \\ &= u_{mn} + \frac{\Delta \tilde{e}_{mn}}{\Delta \tilde{f}_m} - \frac{\Delta y_{mn}}{\Delta x_m}, \end{aligned}$$

where the equality

$$\frac{\Delta e_{mn}}{\Delta f_m} = \frac{\Delta Y_{mn}}{\Delta X_m}$$

was used.

To obtain a discrete approximation of the moving frame constructed in [22], we use the same cross-section replacing derivatives by their finite difference approximations:

$$\begin{aligned} x_{m,n} &= y_{m,n} = u_{m,n} = u_x^{m,n} = u_y^{m,n} = u_{xx}^{m,n} = u_{xy}^{m,n} = 0, \\ u_{yy}^{m,n} &= 1, \quad u_{x^{k+3}}^{m,n} = u_{x^{k+2}y}^{m,n} = \dots = 0, \quad k \geq 0. \end{aligned}$$

The expressions for the discrete derivatives $u_y^{m,n}$, $u_{yy}^{m,n}$, $u_{yyy}^{m,n}$, and $u_{xyy}^{m,n}$ constraint to (3.35) appear in (3.12). Solving the normalization equations

$$X_{m,n} = Y_{m,n} = U_{m,n} = U_X^{m,n} = U_Y^{m,n} = U_{XY}^{m,n} = 0, \quad U_{Y^2}^{m,n} = 1,$$

we obtain the pseudo-group normalizations

$$\begin{aligned} f_m &= 0, \quad e_{m,n} = 0, \quad \frac{\Delta f_m}{\Delta x_m} = \sqrt{u_{yy}^{m,n}}, \quad \frac{\Delta e_{m,n}}{\Delta x_m} = \sqrt{u_{yy}^{m,n}} \left(\frac{\Delta y_{m,n}}{\Delta x_m} - u_{m,n} \right), \\ \frac{\Delta f_{m+1}}{\Delta x_{m+1}} &= \sqrt{u_{yy}^{m,n}} \left(1 - \frac{\Delta x_m \delta u_{m,n}}{\delta u_{m+1,n}} \right), \quad \frac{\Delta e_{m+1}}{\Delta x_{m+1}} = \frac{\Delta f_{m+1}}{\Delta x_{m+1}} \left(\frac{\Delta y_{m+1,n}}{\Delta x_{m+1}} - u_{m+1,n} \right), \\ \frac{\Delta f_{m+2}}{\Delta x_{m+2}} &= \frac{\Delta f_{m+1}}{\Delta x_{m+1}} \left(1 + \frac{\Delta x_{m+1}}{\delta y_{m+2,n}} [\delta y_{m,n} (\Delta y_{m,n} - \Delta x_m u_{m,n}) u_{yy}^{m,n} - u_{m+1,n+1}] \right). \end{aligned}$$

The invariantization map (3.21) provides the normalized joint invariants

$$\begin{aligned} \iota(\Delta x_m) &= \Delta x_m \sqrt{u_{yy}^{m,n}}, \quad \iota(\Delta y_{m,n}) = (\Delta y_{m,n} - u_{m,n} \Delta x_m) \sqrt{u_{yy}^{m,n}}, \\ \iota(\delta y_{m,n}) &= \delta y_{m,n} \sqrt{u_{yy}^{m,n}}, \end{aligned}$$

and

$$I_{03}^d = \iota(u_{yyy}^{m,n}) = \frac{u_{yyy}^{m,n}}{(u_{yy}^{m,n})^{3/2}}, \quad I_{1,2}^d = \iota(u_{xyy}^{m,n}) = \frac{u_{xyy}^{m,n} + u_{m,n} u_{yyy}^{m,n} + 2u_y^{m,n} u_{yy}^{m,n}}{(u_{yy}^{m,n})^{3/2}}. \quad (3.37)$$

In the continuous limit, the joint invariants (3.37) converge to the differential invariants

$$I_{0,3}^d \rightarrow I_{0,3} = \frac{u_{yyy}}{u_{yy}^{3/2}}, \quad I_{1,2}^d \rightarrow I_{1,2} = \frac{u_{xyy} + u u_{yyy} + 2u_y u_{yy}}{u_{yy}^{3/2}},$$

as obtained in [22].

4 Differential and finite difference equations

This section recalls basic definitions pertaining to invariant differential equations and their invariant finite difference approximations, [16, 18]. To treat differential equations and finite difference equations on a similar footing, computational variables are introduced in the continuous setting. Given a differential equation

$$\Delta(x, u^{(n)}) = 0, \quad (4.1)$$

the chain rule (2.10) may be used to re-express (4.1) in terms of $x^i = x^i(s)$, $u^\alpha = u^\alpha(s)$ and their computational derivatives x_{sA}^i , u_{sA}^α :

$$\bar{\Delta}(s, x^{(n)}, u^{(n)}) = \Delta(x, u^{(n)}) = 0, \quad (4.2a)$$

where $(x^{(n)}, u^{(n)}) = (\dots x_{sA}^i \dots u_{sA}^\alpha \dots)$ on the left-hand side of (4.2a) and $u^{(n)} = (\dots u_{xJ}^\alpha \dots)$ on the right-hand side. Equation (4.2a) can be supplemented by *companion equations*, [17],

$$\tilde{\Delta}(s, x^{(n)}, u^{(n)}) = 0, \quad (4.2b)$$

which impose restrictions on the change of variables $s \mapsto x(s)$. For the *extended system* (4.2) to have the same solution space as the original equation (4.1), the companion equations (4.2b) cannot introduce differential constraints on the derivatives u_{sA}^α . Also, they must respect the non-degeneracy condition (2.8).

Definition 4.1. A Lie pseudo-group \mathcal{G} is said to be a *symmetry (pseudo-)group* of a differential equation $\Delta(x, u^{(n)}) = 0$ if for $g \in \mathcal{G}$,

$$\Delta(g \cdot x, g^{(n)} \cdot u^{(n)}) = 0 \quad \text{whenever} \quad \Delta(x, u^{(n)}) = 0.$$

An extended system of differential equations $\{\bar{\Delta}(s, x^{(n)}, u^{(n)}) = 0, \tilde{\Delta}(s, x^{(n)}, u^{(n)}) = 0\}$ is \mathcal{G} -compatible with the \mathcal{G} -invariant differential equation $\Delta(x, u^{(n)}) = 0$ if it is invariant under the pseudo-group \mathcal{G} :

$$\begin{cases} \bar{\Delta}(s, g^{(n)} \cdot x^{(n)}, g^{(n)} \cdot u^{(n)}) = 0, \\ \tilde{\Delta}(s, g^{(n)} \cdot x^{(n)}, g^{(n)} \cdot u^{(n)}) = 0, \end{cases} \quad \text{whenever} \quad \begin{cases} \bar{\Delta}(s, x^{(n)}, u^{(n)}) = 0, \\ \tilde{\Delta}(s, x^{(n)}, u^{(n)}) = 0. \end{cases}$$

Using a perspective slightly different from the one introduced in [2, 16], a numerical scheme for the differential equation (4.1), or its extended counterpart (4.2), is a set of finite difference equations

$$E(\mathfrak{z}_N^{[n]}) = 0, \quad \tilde{E}(\mathfrak{z}_N^{[n]}) = 0,$$

having the property that, in the continuous limit, these equations converge to the extended system (4.2):

$$E(\mathfrak{z}_N^{[n]}) \rightarrow \bar{\Delta}(s, x^{(n)}, u^{(n)}), \quad \tilde{E}(\mathfrak{z}_N^{[n]}) \rightarrow \tilde{\Delta}(s, x^{(n)}, u^{(n)}). \quad (4.3)$$

Definition 4.2. A discretized pseudo-group \mathcal{G}_d is a *symmetry group* of the numerical scheme $\{E(\mathfrak{z}_N^{[n]}) = 0, \tilde{E}(\mathfrak{z}_N^{[n]}) = 0\}$ if

$$\begin{cases} E(g_N^{[n]} \cdot \mathfrak{z}_N^{[n]}) = 0, \\ \tilde{E}(g_N^{[n]} \cdot \mathfrak{z}_N^{[n]}) = 0, \end{cases} \quad \text{whenever} \quad \begin{cases} E(\mathfrak{z}_N^{[n]}) = 0, \\ \tilde{E}(\mathfrak{z}_N^{[n]}) = 0. \end{cases}$$

Given a \mathcal{G} -invariant differential equation $\Delta = 0$, there are many different strategies for constructing an invariant numerical scheme, [1, 12, 16, 20, 26]. Assuming Δ is a differential invariant, one possibility is to obtain an invariant discretization E of Δ using moving frames. This can be done algorithmically by first discretizing Δ to obtain a finite difference approximation F . Since this discretization is not necessarily invariant, see Example 4.3 for an illustration of this fact, an invariant discretization of Δ is obtained by invariantizing F :

$$\Delta \sim E = \iota(F).$$

An invariant approximation of $\Delta = 0$ is then given by $E = 0$. As for the mesh equations $\tilde{E} = 0$, there is, unfortunately, no clear algorithm for determining these equations. Nevertheless, there are obvious requirements that need to be satisfied. First, these equations must include the invariant constraints occurring in the construction of a joint moving frame for the discretized pseudo-group action \mathcal{G}_d . For example, in Example 3.19, the invariant constraint permitting the construction of a joint moving frame is given by

$$x_{m,n+1} - x_{m,n} = 0, \quad (4.4)$$

and this equation would need to be part of the mesh equations of any invariant numerical scheme constructed from the joint moving frame (3.25). For some pseudo-group actions it might be possible to add further invariant mesh equations provided the non-degeneracy constraint (3.4) is satisfied. For example, in Example 3.19, since $Y_{m,n} = y_{m,n}$ is invariant, we assumed that $y_n = k n + y_0$ to simplify computations. Under this assumption, equation (4.4) would be supplemented by the invariant mesh equations

$$y_{m+1,n} - y_{m,n} = 0, \quad y_{m,n+1} - y_{m,n} = k.$$

Example 4.3. A numerical scheme for the differential equation (1.3) invariant under the full (discretized) symmetry pseudo-group (3.30) is constructed. Following the prescription above, the invariant (1.2) is naively discretized on a rectangular mesh

$$I_{1,1} \sim F = \frac{u_{m+1,n+1} u_{m,n} - u_{m+1,n} u_{m,n+1}}{u_{m,n}^3 \Delta x_m \delta y_n}. \quad (4.5)$$

We note that this approximation is not invariant under (3.30). Using the results of Example 3.20, an invariant approximation is obtained by invariantizing (4.5):

$$I_{1,1} \sim I_{1,1}^d = \iota(F) = \frac{u_{m+1,n+1} u_{m,n} - u_{m+1,n} u_{m,n+1}}{u_{m,n} u_{m+1,n} u_{m,n+1} \Delta x_m \delta y_n}.$$

The construction of the joint moving frame in Example 3.20 is based on the assumption that (3.29) holds. Hence, an invariant numerical scheme for (1.3) is given by

$$\frac{u_{m+1,n+1} u_{m,n} - u_{m+1,n} u_{m,n+1}}{u_{m,n} u_{m+1,n} u_{m,n+1} \Delta x_m \delta y_n} = 1, \quad (4.6a)$$

with mesh equations

$$\delta x_{m,n} = 0, \quad \Delta y_{m,n} = 0. \quad (4.6b)$$

The scheme (4.6) is an approximation of the differential equations

$$I_{1,1} = \frac{u u_{st} - u_s u_t}{u^3 x_s y_t} = 1 \quad (4.7a)$$

and

$$x_t = 0, \quad y_s = 0, \quad (4.7b)$$

in the computational variables $x = x(s, t)$, $y = y(s, t)$, $u = u(s, t)$. Equation (4.7a) is simply (1.3) expressed in computational variables while equations (4.7b) are the invariant companion equations of the extended system (4.7).

5 Numerical simulations

In this section, the fully invariant numerical scheme (4.6) is compared with the standard finite difference approximation

$$\begin{aligned} \frac{u_{m+1,n+1} u_{m,n} - u_{m+1,n} u_{m,n+1}}{u_{m,n}^3 \Delta x_m \delta y_n} &= 1, \\ \Delta x_{m,n} &= h, \quad \delta x_{m,n} = 0, \quad \Delta y_{m,n} = 0, \quad \delta y_{m,n} = k \end{aligned} \quad (5.1)$$

of equation (1.3). Since the mesh equations (4.6b) do not specify the step sizes $\Delta x_{m,n}$ and $\delta y_{m,n}$, the equations

$$\Delta x_{m,n} = h, \quad \delta y_{m,n} = k$$

are supplemented to compare the two schemes on the same footing. In other words, the numerical schemes (4.6a) and (5.1) are both defined over the same rectangular mesh.

5.1 Methodology

Equations (5.1) and (4.6a) both relate the values of the solution u at the four corners of a rectangle on the mesh. Given, the value of u at three corners, the equations provide the value of u at the remaining vertex. These equations are suited for initial value problems (IVPs). For example, the value of u in the xy -plane can be calculated if initial conditions on u are specified on two perpendicular axes. Though, in practice, one has to limit itself to a finite rectangular domain and the specification of u on two of its sides will completely determine the solution on the rectangle. Figure 4 illustrates the situation on a 4×4 rectangle. At each step the value of u at the blue dot is a function of the solution at the green dots. Filling the rectangle from left to right and then from bottom up, the whole rectangle is covered.

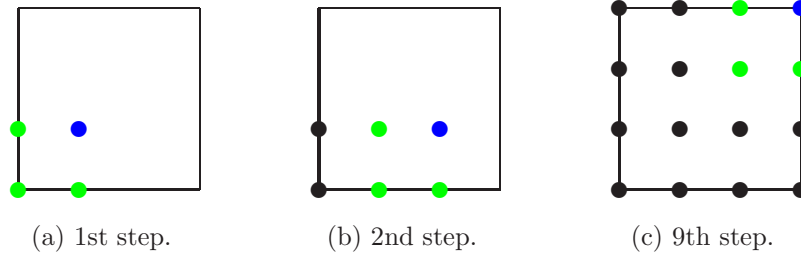


Figure 4: Initial value problem on a rectangle.

On the other hand, numerical schemes like (5.1) and (4.6a) are ill-defined for boundary value problems (BVP) on rectangular domains. Figure 5 illustrates the issue. If, for example, one starts the iterative process in the bottom left corner of the domain of integration, then all points on the right and top boundaries highlighted in red in Figure 5b are ill-defined since their values are simultaneously specified by the boundary conditions and the numerical scheme.

Since in Section 5.2 we are interested in solving BVPs numerically, we now explain how to adapt the schemes (5.1) and (4.6a) to BVPs on rectangular domains. For this, we note that each point in the interior domain can be computed in four different ways using the numerical schemes. First, solving for $u_{m+1,n+1}$ in the invariant scheme (4.6) we obtain

$$u_{m+1,n+1} = u_{m+1,n} u_{m,n+1} \left(\frac{1}{u_{m,n}} + h k \right). \quad (5.2)$$

Then, shifting (4.6) from (m, n) to $(m+1, n)$, the solution $u_{m+1,n+1}$ can also be expressed as

$$u_{m+1,n+1} = \frac{u_{m+1,n} u_{m+2,n+1}}{u_{m+2,n} (1 + h k u_{m+1,n})}. \quad (5.3)$$

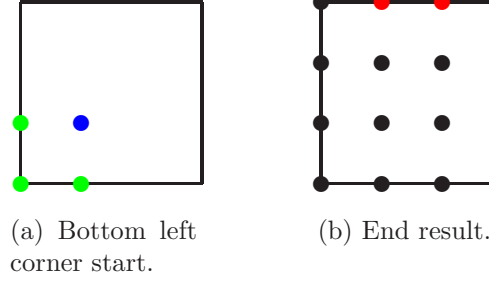


Figure 5: Ill-defined boundary value problem on a rectangle.

Similarly, shifting the invariant scheme (4.6) from (m, n) to $(m, n+1)$ and $(m+1, n+1)$ we obtain

$$u_{m+1,n+1} = \frac{u_{m,n+1} u_{m+1,n+2}}{u_{m,n+2}(1 + h k u_{m,n+1})}, \quad u_{m+1,n+1} = \frac{u_{m+1,n} u_{m+2,n+1}}{u_{m+2,n}(1 + h k u_{m+1,n})}. \quad (5.4)$$

Defining $u_{m+1,n+1}$ to be the average of the four equations (5.2), (5.3), (5.4) yields a finite difference equation expressing each interior point in the domain as a function of its eight surrounding points as illustrated in Figure 6. The same procedure applies to the standard scheme (5.1). The two new schemes are now well-adapted to BVPs on rectangular domains since there is no conflict between the points computed using the numerical schemes and the boundary conditions.

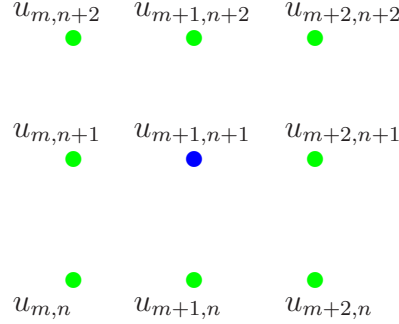


Figure 6: New scheme on nine points. The value of u at the blue dot is determined by the neighbouring green points.

Solutions to BVPs are then obtained by applying the *relaxation method*. The first step in the implementation of the relaxation method consists of assigning values to the points inside the domain of integration. In principle, arbitrary values can be assigned but it is always advantageous to assign well-educated initial values. In our case, we decided to use the average of the four solutions obtained by solving the IVPs starting in each corner of the rectangular domain. Once this is done, new values are assigned to the interior points using the BVP adapted scheme. Figure 7 illustrates the order in which one could assign these new interior values on a 5×5 square. Recomputing the interior values once using the most recent data is one iteration of the relaxation process. If the scheme is stable, by iterating the relaxation process the interior values will converge towards the scheme's solution.

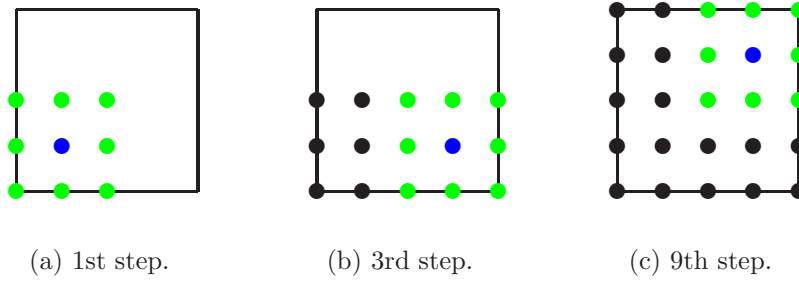


Figure 7: Scheme on nine points covering a rectangular BVP.

5.2 Numerical results and analysis

Three BVPs were tested using the exact solutions

$$u = \frac{2}{(x+y)^2}, \quad u = 2 \sec^2(x+y), \quad u = \frac{2e^{x+y}}{(e^{x+y} - 1)^2}, \quad (5.5)$$

obtained in [25]. In each case, the boundary condition is given by the value of the exact solution on the edges of a rectangular domain. We note that the first and third solutions are not defined along the line $y + x = 0$ and diverge to infinity on both sides of the singular line. The second solution also diverges along the lines $y + x = \pi/2 + n\pi$, with $n \in \mathbb{Z}$. Since the quantitative results are similar for each solution, only the secant solution is presented below.

Table 1 lists the average error of the invariant and standard schemes (4.6) and (5.1) for different values of h and k for the secant solution on the unit square $[1, 2] \times [1, 2]$ after 100 iterations of the relaxation procedure. For the cases considered, the invariant scheme is roughly three times more precise than the standard scheme.

Scheme	$h, k = 0.1$	$h, k = 0.05$	$h, k = 0.01$	$h, k = 0.005$
Standard	2.19×10^{-1}	1.07×10^{-1}	3.23×10^{-2}	1.66×10^{-2}
Invariant	4.12×10^{-2}	2.75×10^{-2}	1.03×10^{-2}	5.42×10^{-3}

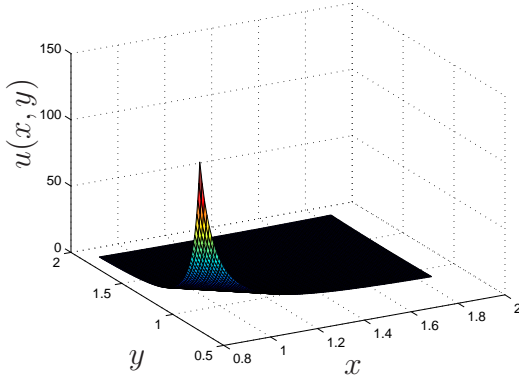
Table 1: Average errors on $[1, 2] \times [1, 2]$ for the secant solution after 100 relaxation iterations.

As demonstrated in [2, 12], invariant schemes seem to shine near singularities. Here again the invariant scheme is more precise and stable near singularities. Table 2 shows the maximal error for both methods when the bottom left corner of the unit square of integration is brought closer to the exact solution singularity at $(\pi/4, \pi/4) \approx (0.785, 0.785)$. The first row in the table gives the coordinates (x_0, y_0) of the square of integration's bottom left corner. The step size in the independent variables is set to $h = k = 0.01$, and the relaxation process was again run a hundred times. As the square of integration gets closer to the singularity, Table 2 shows that the precision of the standard method gets worst much faster than the invariant scheme. Moreover, when $x_0 = y_0 = 0.84$ or anywhere closer to the singularity $(\pi/4, \pi/4)$, the standard scheme becomes unstable while the invariant method integrates further into

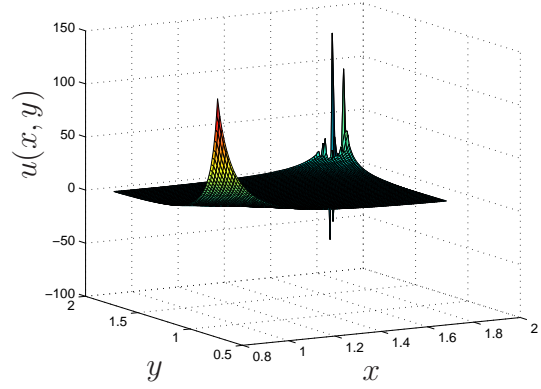
the singularity. As shown in Figure 8b, while the source of the instability is in the bottom left corner, its manifestation appears first in the opposite corner for the standard method. Meanwhile, the invariant method, Figure 8c, is faithful to the exact solution, Figure 8a.

Scheme	$x_0 = y_0 = 0.87$	0.86	0.85	0.84
Standard	2.20	4.92	129.03	unstable
Invariant	3.12×10^{-1}	4.46×10^{-1}	6.78×10^{-1}	1.15

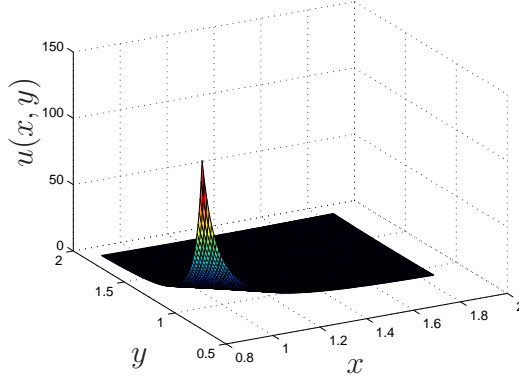
Table 2: Maximal errors on a unit square near the singularity $(\pi/4, \pi/4)$.



(a) Exact solution.



(b) Standard scheme.



(c) Invariant scheme.

Figure 8: Secant solution near the singularity $(\pi/4, \pi/4)$.

It is not difficult to understand why the invariant scheme produces better results when compared to the standard scheme. The distinctive feature between the two schemes is the way the cubic term u^3 in (1.3) is approximated. In the naive discretization (5.1), u^3 is approximated by the nonlinear term $u_{m,n}^3$. This cubic term in the standard scheme requires the use of a nonlinear equation solver like Newton's method

at each iteration of the relaxation method which increases the computational cost and adds instability. On the other hand, in the invariant scheme the cubic term u^3 is approximated by $u_{m,n} u_{m+1,n} u_{m,n+1}$. By using the values of u at three distinct points, the invariant method is more precise and stable, especially where the solution varies a lot. Moreover, (4.6) can be solved for any of the u 's without the need to resort to a nonlinear solver. Thanks to this simplification, the computation time for the invariant method was approximately three times shorter than that of the standard method in all our numerical simulations.

As previously mentioned, similar results were also obtained for the rational and exponential solutions of (5.5).

6 Conclusion

To the best of our knowledge, this is the first work attempting to construct invariant numerical schemes of differential equations with infinite-dimensional symmetry groups. As our examples show, the main issue with considering the product action of Lie pseudo-groups is the shortage of joint invariants to approximate differential invariants. To circumvent this problem, we proposed to discretize the action by replacing derivatives with finite difference approximations. To illustrate our constructions as clearly as possible, we chose simple Lie pseudo-group actions that have been well studied in the continuous setting. The next natural step in this line of research would be to consider more substantial symmetry pseudo-groups and apply Lie theoretical tools to these invariant schemes to find explicit solutions.

The main emphasis of the paper was on the theoretical aspects that emerge when infinite-dimensional symmetry groups are discretized. Although the numerical simulations performed in Section 5 do not have the pretension to be the state of the art in numerical analysis, they indicate that invariant schemes can produce good numerical results. It remains a challenge to bridge the gap between the most recent trends in numerical analysis and the latest developments in the theory of invariant finite difference equations.

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